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The solutions of the NLS equations with self-consistent sources

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Abstract

We construct the generalized Darboux transformation with arbitrary functions at time t for the AKNS equation with self-consistent sources (AKNSESCS) which, in contrast with the Darboux transformation for the AKNS equation, provides a non-auto-Bäcklund transformation between two AKNSESCSs with different degrees of sources. The formula for N-times repeated generalized Darboux transformation is proposed. By reduction the generalized Darboux transformation with arbitrary functions at time t for the nonlinear Schrödinger equation with self-consistent sources (NLSESCS) is obtained and enables us to find the dark soliton, bright soliton and positon solutions for NLS$^+$ESCS and NLS$^-$ESCS. The properties of these solution are analysed.

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1. Introduction

The nonlinear Schrödinger equation with self-consistent sources (NLSESCS) describes the soliton propagation in a medium with both resonant and nonresonant nonlinearities [1–4], and it also describes the nonlinear interaction of high-frequency electrostatic waves with ion acoustic waves in plasma [5]. Some soliton solution for the NLSESCS was obtained by inverse scattering transformation in [1]. Since the explicit time part of the Lax representation of the NLSESCS was not found, the method to solve the NLSESCS by inverse scattering transformation in [1] was quite complicated.

Due to the important role played by the soliton equations with self-consistent sources (SESCSs) in many fields of physics, such as hydrodynamics, solid state physics, plasma physics, SESCSs have attracted some attention [6–16]. In recent years we have presented a method to find the explicit time part of the Lax representation for SESCSs and to construct generalized binary Darboux transformations with arbitrary functions at time t for SESCSs which, in contrast with the Darboux transformation for soliton equations [17, 18], offer a non-auto-Bäcklund transformation between two SESCSs with different degrees of sources and can be used to obtain N-soliton, positon and negaton solutions [19–21].

The positon solution for many soliton equations and their physical application have been widely studied, for example, the positon solutions for KdV and mKdV equations were investigated in [23, 24], for the nonlinear Schrödinger equation in [25], for the sine-Gordon equation in [26]. However positon solutions for SESCSs except for the KdV equation with self-consistent sources in [19, 20] have not been studied.

In this paper, we develop the method presented in [19, 20] to study the NLSESCS. First we construct the generalized Darboux transformation with arbitrary functions at time t for the AKNS equation with self-consistent sources (AKNSESCS) which offers a non-auto-Bäcklund transformation between two AKNSESCSs with different degrees of sources. Then by reduction we obtained the generalized Darboux transformation with arbitrary functions at time t for the NLSESCS which also provides a non-auto-Bäcklund transformation between two NLSESCSs with different degrees of sources. Some interesting solutions of NLSESCS such as dark soliton, bright soliton and positon solutions for NLS$^+$ESCS and NLS$^-$ESCS are found. The properties of these solutions are analysed.

2. Binary Darboux transformations for the AKNS equation with self-consistent sources

The AKNSESCS is defined as [15, 16]

$$q_t = -\mathrm{i}(q_{xx} - 2q^2 r) + \sum_{j=1}^{n} \left(\varphi_j^{(1)}\right)^2, \qquad r_t = \mathrm{i}(r_{xx} - 2qr^2) + \sum_{j=1}^{n} \left(\varphi_j^{(2)}\right)^2, \tag{2.1a}$$

$$\varphi_{j,x} = \begin{pmatrix} -\lambda_j & q \\ r & \lambda_j \end{pmatrix} \varphi_j, \qquad j = 1, \ldots, n, \tag{2.1b}$$

where λ_j are n distinct complex constants, $\varphi_j = \left(\varphi_j^{(1)}, \varphi_j^{(2)}\right)^T$ (hereafter, we use superscripts (1) and (2) to denote the first and second elements of a two-dimensional vector respectively).

The Lax pair for equations (2.1) is given by [15, 16]

$$\psi_x = U\psi, \qquad U := U(\lambda, q, r) = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \tag{2.2a}$$

$$\psi_{t_s} = R^{(n)}\psi, \qquad R^{(n)} := V + \sum_{j=1}^{n} \frac{H(\varphi_j)}{\lambda - \lambda_j}, \tag{2.2b}$$

where

$$V := V(\lambda, q, r) = \mathrm{i} \begin{pmatrix} -2\lambda^2 + qr & 2\lambda q - q_x \\ 2\lambda r + r_x & 2\lambda^2 - qr \end{pmatrix}, \qquad H(\varphi_j) = \frac{1}{2} \begin{pmatrix} -\varphi_j^{(1)}\varphi_j^{(2)} & \left(\varphi_j^{(1)}\right)^2 \\ -\left(\varphi_j^{(2)}\right)^2 & \varphi_j^{(1)}\varphi_j^{(2)} \end{pmatrix}.$$

2.1. Binary Darboux transformation with an arbitrary constant

It is known [16] that based on the Darboux transformation for the AKNS equation [22], the AKNSESCS admits two elementary Darboux transformations $\mathcal{T}_{1,2} : (q, r, \varphi_1, \ldots, \varphi_n) \mapsto (\widetilde{q}, \widetilde{r}, \widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$. Given two arbitrary complex numbers μ and ν, $\mu \neq \nu$, let $f = f(\mu)$ and $g = g(\nu)$ be two solutions of (2.2) with $\lambda = \mu$ and $\lambda = \nu$ respectively, and define $\mathcal{T}_1[f]$:

$$\widetilde{\psi} = T_1\psi, \qquad T_1 = T_1(\lambda, f) = \begin{pmatrix} \lambda - \mu + qf^{(2)}/(2f^{(1)}) & -q/2 \\ -f^{(2)}/f^{(1)} & 1 \end{pmatrix},$$

$$\widetilde{q} = -q_x/2 - \mu q + q^2 f^{(2)}/(2f^{(1)}), \qquad \widetilde{r} = 2f^{(2)}/f^{(1)},$$

$$\widetilde{\varphi}_j = \frac{T_1(\lambda_j, f)\varphi_j}{\sqrt{\lambda_j - \mu}}, \qquad j = 1, \ldots, n,$$

$\mathcal{T}_2[g]$:

$$\widetilde{\psi} = T_2\psi, \qquad T_2 = T_2(\lambda, g) = \begin{pmatrix} 1 & -g^{(1)}/g^{(2)} \\ r/2 & \lambda - \nu - rg^{(1)}/(2g^{(2)}) \end{pmatrix},$$

$$\widetilde{q} = -2g^{(1)}/g^{(2)}, \qquad \widetilde{r} = r_x/2 - \nu r - r^2 g^{(1)}/(2g^{(2)}),$$

$$\widetilde{\varphi}_j = \frac{T_2(\lambda_j, g)\varphi_j}{\sqrt{\lambda_j - \nu}}, \qquad j = 1, \ldots, n.$$

Theorem 2.1. *The linear system (2.2) is covariant with respect to (wrt) the two Darboux transformations $\mathcal{T}_1, \mathcal{T}_2$, i.e., the new variables $\widetilde{\psi}, \widetilde{q}, \widetilde{r}$ and $\widetilde{\varphi}_j$ satisfy*

$$\widetilde{\psi}_x = \widetilde{U}\widetilde{\psi}, \qquad \widetilde{U} = U(\lambda, \widetilde{q}, \widetilde{r}), \tag{2.3a}$$

$$\widetilde{\psi}_t = \widetilde{R}^{(n)}\widetilde{\psi} := \left[V^{(s)}(\lambda, \widetilde{q}, \widetilde{r}) + \sum_{j=1}^{n} \frac{H(\widetilde{\varphi}_j)}{\lambda - \lambda_j} \right] \widetilde{\psi}. \tag{2.3b}$$

We now construct a new Darboux transformation based on $\mathcal{T}_1$ and $\mathcal{T}_2$. Our method is similar to that for the KdV equation with self-consistent sources [20]. Define

$$\sigma(f, g) := -\frac{W(f, g)}{2(\mu - \nu)}, \qquad \sigma(f, f) := \lim_{\lambda \to \mu} \frac{-W(f(\lambda), f(\mu))}{2(\lambda - \mu)} = \frac{1}{2} W(f, \partial_\mu f),$$

where $W(f, g)$ is the Wronskian $W(f, g) := f^{(1)}g^{(2)} - f^{(2)}g^{(1)}$. We assume that we have obtained $(\widetilde{\psi}, \widetilde{q}, \widetilde{r}, \widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$ satisfying (2.3) by applying $\mathcal{T}_1[f]$ to $(\psi, q, r, \varphi_1, \ldots, \varphi_n)$. Then we derive two linearly independent solutions of (2.3) with $\lambda = \mu$ and in terms of f only.

First solution. Let $f_1 = f_1(\mu)$ be a solution of (2.2) with $\lambda = \mu$, and $W(f, f_1) \neq 0$ (i.e., f and f_1 are linearly independent). Then applying $\mathcal{T}_1[f]$ to f_1 gives a solution of (2.3) with $\lambda = \mu$:

$$\widetilde{f}_1 := T_1(\mu, f)f_1 = \frac{W(f, f_1)}{2f^{(1)}} \begin{pmatrix} -q \\ 2 \end{pmatrix}.$$

Since $W(f, f_1)$ is independent of both x and t, we assume $W(f, f_1) \equiv 1$. Thus, we obtain the first solution of (2.3):

$$\widetilde{f}_1 = \frac{1}{2f^{(1)}} \begin{pmatrix} -q \\ 2 \end{pmatrix}.$$

Second solution. Note that $\psi_1(\lambda) := f(\lambda)/(\lambda - \mu)$ is a solution of (2.2). Applying $\mathcal{T}_1[f]$ to ψ_1 gives a solution of (2.3):

$$\widetilde{\psi}_1(\lambda) = T_1(\lambda, f)\psi_1 = \begin{pmatrix} f^{(1)}(\lambda) \\ 0 \end{pmatrix} + \frac{W(f(\mu), f(\lambda))}{2f^{(1)}(\mu)(\lambda - \mu)} \begin{pmatrix} -q \\ 2 \end{pmatrix}.$$

Taking the limit, we find a second solution of (2.3) with $\lambda = \mu$:

$$\widetilde{f} := \lim_{\lambda \to \mu} \widetilde{\psi}_1(\lambda) = \begin{pmatrix} f^{(1)} \\ 0 \end{pmatrix} + \frac{\sigma(f, f)}{f^{(1)}} \begin{pmatrix} -q \\ 2 \end{pmatrix}.$$

Let C be an arbitrary constant, then the linear combination of the above solutions

$$\widetilde{h} := \widetilde{f} + 2C\widetilde{f}_1 = \begin{pmatrix} f^{(1)} \\ 0 \end{pmatrix} + \frac{C + \sigma(f, f)}{f^{(1)}} \begin{pmatrix} -q \\ 2 \end{pmatrix}$$

is also a solution of (2.3) with $\lambda = \mu$. Apply $\mathcal{T}_2[\widetilde{h}]$ to $(\widetilde{\psi}/(\lambda - \mu), \widetilde{q}, \widetilde{r}, \widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$, i.e., define

$$\widehat{\psi} = T_2(\lambda, \widetilde{h})\frac{\widetilde{\psi}}{\lambda - \mu} = \psi - \frac{f}{C + \sigma(f, f)}\sigma(f, \psi), \tag{2.4a}$$

$$\widehat{q} = -\frac{\widetilde{h}_1}{\widetilde{h}_2} = q - \frac{2(f^{(1)})^2}{C + \sigma(f, f)}, \qquad \widehat{r} = \frac{\widetilde{r}_x}{2} - \mu\widetilde{r} - \frac{r^2\widetilde{h}_1}{2\widetilde{h}_2} = r - \frac{(f^{(2)})^2}{C + \sigma(f, f)}, \tag{2.4b}$$

$$\widehat{\varphi}_j = \frac{T_2(\lambda_j, \widetilde{h})\widetilde{\varphi}_j}{\sqrt{\lambda_j - \mu}} = \varphi_j - \frac{f}{C + \sigma(f, f)}\sigma(f, \varphi_j), \tag{2.4c}$$

then the new variables $\widehat{\psi}, \widehat{q}, \widehat{r}, \widehat{\varphi}_j$ satisfy

$$\widehat{\psi}_x = \widehat{U}\widehat{\psi}, \tag{2.5a}$$

$$\widehat{\psi}_t = \widehat{R}^{(n)}\widehat{\psi}, \tag{2.5b}$$

where

$$\widehat{U} = U(\lambda, \widehat{q}, \widehat{r}) \qquad \text{and} \qquad \widehat{R}^{(n)} = V(\lambda, \widehat{q}, \widehat{r}) + \sum_{j=1}^{n} H(\widehat{\varphi}_j)/(\lambda - \lambda_j).$$

Proposition 2.1. *Let f be a solution of (2.2) with $\lambda = \mu$, and C be an arbitrary constant, then $\widehat{\psi}, \widehat{q}, \widehat{r}$ and $\widehat{\varphi}_j$ given by (2.4) present a binary Darboux transformation with an arbitrary constant for (2.2), and $(\widehat{q}, \widehat{r}, \widehat{\varphi}_1, \ldots, \widehat{\varphi}_n)$ is a new solution of (2.1). Moreover, we have*

$$\widehat{q}\,\widehat{r} = qr - \partial_x^2 \log[C + \sigma(f, f)].$$

2.2. Binary Darboux transformation with an arbitrary function of t

Substituting (2.4a) into the left-hand side of equation (2.5b), we have a polynomial of $[C + \sigma(f, f)]^{-1}$:

$$\begin{aligned}
\widehat{\psi}_t &= \frac{\partial}{\partial t}\left[\psi - \frac{f}{C + \sigma(f, f)}\sigma(f, \psi)\right] \\
&= \psi_t - \frac{f_t}{C + \sigma(f, f)}\sigma(f, \psi) - \frac{f[W(f_t, \psi) + W(f, \psi_t)]}{2(\mu - \lambda)[C + \sigma(f, f)]} \\
&\quad + \frac{f\sigma(f, \psi)[W(f_t, f_\mu) + W(f, f_{t,\mu})]}{2[C + \sigma(f, f)]^2} =: \sum_{j=0}^{2} L_j[C + \sigma(f, f)]^{-j},
\end{aligned}$$

where L_j are two-dimensional vector functions defined by the last equality. We can expect that substituting (2.4) into the right-hand side of (2.5b) will also give a polynomial of $[C + \sigma(f, f)]^{-1}$, but it will be more complicated. So we just write it as

$$\widehat{R}^{(n)}\widehat{\psi} = \sum_{j=0}^{3} R_j[C + \sigma(f, f)]^{-j},$$

where R_j are also two-dimensional vector functions dependent on ψ, q, r, φ_j and f and their derivatives wrt x. Since (2.5b) holds for any constant C, we have the following lemma.

Lemma 2.1. *Assume that ψ, q, r and φ_j satisfy (2.2), and let f be a solution of (2.2) with $\lambda = \mu$, then we have*

$$L_j = R_j, \qquad j = 0, 1, 2, \qquad R_3 = 0,$$

for all x and t.

We now replace the constant C with an arbitrary function of t, say $c(t)$. Since there is no derivatives wrt t in the expression of $\widehat{R}^{(n)}$, if we replace C with $c(t)$ in the definition of (2.4), we will have

$$\widehat{R}^{(n)}\widehat{\psi} = \sum_{j=0}^{3} R_j[c(t) + \sigma(f, f)]^{-j}.$$

But we will not have $\widehat{\psi}_t = \sum_{j=0}^{3} L_j[c(t) + \sigma(f, f)]^{-j}$ under this replacement. However, this replacement will lead to a non-auto-Bäcklund transformation.

Proposition 2.2. *Let f be a solution of (2.2) with $\lambda = \lambda_{n+1}$, and $c(t)$ be an arbitrary function of t. If we define*

$$\bar{\psi} = \psi - \frac{f}{c(t) + \sigma(f, f)}\sigma(f, \psi), \tag{2.6a}$$

$$\bar{q} = q - \frac{(f^{(1)})^2}{c(t) + \sigma(f, f)}, \qquad \bar{r} = r - \frac{(f^{(2)})^2}{c(t) + \sigma(f, f)}, \tag{2.6b}$$

$$\bar{\varphi}_j = \varphi_j - \frac{f}{c(t) + \sigma(f, f)}\sigma(f, \varphi_j), \qquad j = 1, \ldots, n, \tag{2.6c}$$

and

$$\bar{\varphi}_{n+1} = \frac{\sqrt{\dot{c}(t)}f}{c(t) + \sigma(f, f)}\sigma(f, \varphi_j), \tag{2.6d}$$

then the new variables $\bar{\psi}, \bar{q}, \bar{r}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{n+1}$ satisfy a new system

$$\bar{\psi}_x = \bar{U}\bar{\psi}, \qquad \bar{U} = U(\lambda, \bar{q}, \bar{r}), \tag{2.7a}$$

$$\bar{\psi}_t = \bar{R}^{(n+1)}\bar{\psi}, \qquad \bar{R}^{(n+1)} = V(\lambda, \bar{q}, \bar{r}) + \sum_{j=1}^{n+1} \frac{H(\bar{\varphi}_j)}{\lambda - \lambda_j}, \tag{2.7b}$$

and $(\bar{q}, \bar{r}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{n+1})$ is a solution of (2.1) with n replaced by $n + 1$. Moreover, we have

$$\bar{q}\bar{r} = qr - \partial_x^2 \log[c(t) + \sigma(f, f)].$$

Proof. Since no derivatives wrt t appear in equation (2.7a), it is covariant wrt the transformation defined by (2.6). Substitution of (2.6a) into the left side of (2.7b) gives

$$\begin{aligned}
\bar{\psi}_t &= \frac{\partial}{\partial t}\left[\psi - \frac{f}{c(t) + \sigma(f, f)}\sigma(f, \psi)\right] = \psi_t - \frac{f_t}{c(t) + \sigma(f, f)}\sigma(f, \psi) \\
&\quad - \frac{f[W(f_t, \psi) + W(f, \psi_t)]}{2(\mu - \lambda)[c(t) + \sigma(f, f)]} + \frac{f\sigma(f, \psi)[2\dot{c}(t) + W(f_t, f_\mu) + W(f, f_{t,\mu})]}{2[c(t) + \sigma(f, f)]^2} \\
&= \sum_{j=0}^{2} L_j[c(t) + \sigma(f, f)]^{-j} + \frac{\dot{c}(t)f\sigma(f, \psi)}{[c(t) + \sigma(f, f)]^2}
\end{aligned}$$

$$= \sum_{j=0}^{2} R_j[c(t)+\sigma(f,f)]^{-j} + \frac{\sqrt{\dot{c}(t)}\sigma(f,\psi)}{c(t)+\sigma(f,f)}\bar{\varphi}^{n+1}$$

$$= \bar{R}^{(n)}\bar{\psi} + \frac{H(\bar{\varphi}_{n+1})}{2(\lambda-\lambda_{n+1})}\bar{\psi} = \bar{R}^{(n+1)}\bar{\psi}.$$

This completes the proof. □

Example of solution. We start from equations (2.1) with $n = 0$, and the initial solution $q = r = 0$. Choose a solution of (2.2) with $n = 0, q = r = 0$ as $f = \left(e^{-\lambda_1 x - 2i\lambda_1^2 t}, e^{\lambda_1 x + 2i\lambda_1^2 t}\right)^T$, then by proposition 2.2, we obtain a solution of (2.1) with $n = 1$:

$$q = -\frac{e^{-2\lambda_1 x - 4i\lambda_1^2 t}}{x + 4i\lambda_1 t + c(t)}, \qquad r = -\frac{e^{2\lambda_1 x + 4i\lambda_1^2 t}}{x + 4i\lambda_1 t + c(t)},$$

$$\varphi_1 = \frac{\sqrt{\dot{c}(t)}}{x + 4i\lambda_1 t + c(t)}\begin{pmatrix} e^{-\lambda_1 x - 2i\lambda_1^2 t} \\ e^{\lambda_1 x + 2i\lambda_1^2 t}\end{pmatrix},$$

where $c(t)$ is an arbitrary function.

Remark. The binary Darboux transformation (2.6), in fact, provides a non-auto-Bäcklund transformation between the AKNS equation with sources of different degrees of freedom. Since a function $c(t)$ is involved, we call it a binary Darboux transformation with an arbitrary function of t. This transformation is dependent on two elements, $c(t)$ and f, so we just write them together as a pair $\{c, f\}$.

2.3. Multi-times repeated binary Darboux transformation with arbitrary functions

It is evident that the binary Darboux transformation with an arbitrary function can be applied N times, and we will obtain the N-times repeated binary Darboux transformtion with N arbitrary functions. Let $f_1, f_2, \ldots,$ be a series of solutions of (2.2) with $\lambda = \lambda_1, \lambda_2, \ldots,$ and let $c_1, c_2, \ldots,$ be a series of arbitrary functions of t. Let $\psi[N], q[N], r[N], \varphi_j[N]$ and $f_j[N]$ denote the N-times Darboux transformed variables.

We define some symmetric forms. Let c_j and g_j, $j = 1, 2, \ldots$ be a series of scalar and two-dimensional vectors, u be a scalar, h be a two-dimensional vector, and $\sigma(g_i, g_j)$ and $\sigma(g_i, h)$ are defined. For $N = 1, 2, \ldots,$ we define five forms W_0, $W_1^{(i)}$ and $W_2^{(i)}$, $i = 1, 2$, which are symmetric for the N pairs $\{c_j, g_j\}$, as follows:

$$W_0(\{c_1, g_1\}, \ldots, \{c_N, g_N\}) = \det A,$$

$$W_1^{(i)}(\{c_1, g_1\}, \ldots, \{c_N, g_N\}; h) = \det\begin{pmatrix} A & b \\ \alpha^{(i)} & h^{(i)}, \quad i = 1, 2, \end{pmatrix}$$

$$W_2^{(i)}(\{c_1, g_1\}, \ldots, \{c_N, g_N\}; u) = \det\begin{pmatrix} A & (\alpha^{(i)})^T \\ \alpha^{(i)} & u, \quad i = 1, 2, \end{pmatrix}$$

where

$$A = (\delta_{ij}c_i + \sigma(g_i, g_j))_{N\times N}, \qquad b = (\sigma(g_1, h), \ldots, \sigma(g_N, h))^T, \qquad \alpha^{(i)} = \left(g_1^{(i)}, \ldots, g_N^{(i)}\right).$$

For convenience, we define

$$W_1(\{c_1, g_1\}, \ldots, \{c_N, g_N\}; h) = \begin{pmatrix} W_1^{(1)}(\{c_1, g_1\}, \ldots, \{c_N, g_N\}; h) \\ W_1^{(2)}(\{c_1, g_1\}, \ldots, \{c_N, g_N\}; h)\end{pmatrix}.$$

Lemma 2.2. *Let* $F_i[j] = \{c_i, f_i[j]\}$, $i, j = 1, 2, \ldots$, *then for* $l, k = 1, 2, \ldots$, *we have*

$$W_0(F_{l+1}[l], \ldots, F_{l+k}[l]) = \frac{W_0(F_l[l-1], \ldots, F_{l+k}[l-1])}{W_0(F_l[l-1])} \tag{2.8a}$$

$$W_1(F_{l+1}[l], \ldots, F_{l+k}[l]; \psi[l]) = \frac{W_1(F_l[l-1], \ldots, F_{l+k}[l-1]; \psi[l-1])}{W_0(F_l[l-1])}, \tag{2.8b}$$

$$W_2^{(1)}(F_{l+1}[l], \ldots, F_{l+k}[l]; q[l]) = \frac{W_2^{(1)}(F_l[l-1], \ldots, F_{l+k}[l-1]; q[l-1])}{W_0(F_l[l-1])}, \tag{2.8c}$$

$$W_2^{(2)}(F_{l+1}[l], \ldots, F_{l+k}[l]; r[l]) = \frac{W_2^{(2)}(F_l[l-1], \ldots, F_{l+k}[l-1]; r[l-1])}{W_0(F_l[l-1])}, \tag{2.8d}$$

Proof. Let $a_{ij} = \delta_{ij} c_{l+i} + \sigma(f_{l+i}[l-1], f_{l+j}[l-1])$, $i, j = 1, 2, \ldots$. Direct calculation yields

$$\delta_{ij} c_{l+i} + \sigma(f_{l+i}[l], f_{l+j}[l]) = a_{ij} - a_{i0} a_{00}^{-1} a_{0j} \equiv \bar{a}_{ij}, \qquad i, j = 1, 2, \ldots.$$

Note that

$$\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0k} \\ a_{10} & a_{11} & \cdots & a_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k0} & a_{k1} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} 1 & -a_{00}^{-1} a_{01} & \cdots & -a_{00}^{-1} a_{0k} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_{00} & 0 & \cdots & 0 \\ a_{10} & \bar{a}_{11} & \cdots & \bar{a}_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k0} & \bar{a}_{k1} & \cdots & \bar{a}_{kk} \end{pmatrix}.$$

Taking the determinant for both sides, we have

$$W_0(F_l[l-1], \ldots, F_{l+k}[l-1])) = W_0(F_l[l-1]) W_0(F_{l+1}[l], \ldots, F_{l+k}[l]),$$

which is just equation (2.8a). Similarly, we can prove (2.8b), (2.8c) and (2.8d). □

Proposition 2.3. *For* $N = 1, 2, 3, \ldots$, *we have*

$$\psi[N] = \frac{1}{\Delta} W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; \psi), \tag{2.9a}$$

$$q[N] = \frac{1}{\Delta} W_2^{(1)}(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; q), \tag{2.9b}$$

$$r[N] = \frac{1}{\Delta} W_2^{(2)}(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; r), \tag{2.9c}$$

$$\varphi_j[N] = \frac{1}{\Delta} W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; \varphi_j), \qquad j = 1, \ldots, n, \tag{2.9d}$$

$$\varphi_{n+j}[N] = \frac{\sqrt{\dot{c}_j}}{c_j \Delta} W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; f_j), \qquad j = 1, \ldots, N, \tag{2.9e}$$

and

$$q[N] r[N] = qr - \partial_x^2 \log \Delta \tag{2.9f}$$

where $\Delta = W_0(\{c_1, f_1\}, \ldots, \{c_N, f_N\})$.

Proof. By the definition of $\psi[N]$ and lemma 2.2, we have

$$\begin{aligned} \psi[N] &= \frac{W_1(\{c_N, f_N[N-1]\}; \psi[N-1])}{W_0(\{c_N, f_N[N-1]\})} \\ &= \frac{W_1(\{c_{N-1}, f_{N-1}[N-2]\}, \{c_N, f_N[N-2]\}; \psi[N-2])}{W_0(\{c_{N-1}, f_{N-1}[N-2]\})} \\ &\quad \times \frac{W_0(\{c_{N-1}, f_{N-1}[N-2]\})}{W_0(\{c_{N-1}, f_{N-1}[N-2]\}, \{c_N, f_N[N-2]\})} \\ &= \cdots = \frac{W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; \psi)}{W_0(\{c_1, f_1\}, \ldots, \{c_N, f_N\})}, \end{aligned}$$

which gives rise to equation (2.9*a*). Similarly, we can prove (2.9*b*), (2.9*c*), (2.9*d*) and (2.9*e*). □

3. Binary Darboux transformations for the NLS equations with self-consistent sources

It is well known that from the ordinary AKNS equation

$$q_t = -\mathrm{i}(q_{xx} - 2q^2 r), \qquad r_t = \mathrm{i}(r_{xx} - 2qr^2). \tag{3.1}$$

if we set $r = \varepsilon q^*$, $\varepsilon = \pm 1$, then equations (3.1) are reduced to the ordinary NLS eqution

$$q_t = \mathrm{i}(2\varepsilon |q|^2 q - q_{xx}). \tag{3.2}$$

We call the equation with $\varepsilon = +1$ the NLS$^+$ equation and the equation with $\varepsilon = -1$ the NLS$^-$ equation.

Similarly, we can reduce the AKNSESCS into the NLS$^\pm$ equations with self-consistent sources (NLS$^\pm$ ESCS), but the reductions are more complicated since the sources need to be reduced as well. First, we define two linear maps S_+ and S_- by

$$S_\pm : \begin{pmatrix} z^{(1)} \\ z^{(2)} \end{pmatrix} \mapsto \begin{pmatrix} \pm z^{(2)*} \\ z^{(1)*} \end{pmatrix}. \tag{3.3}$$

For the reduced AKNS spectral problem, i.e., the NLS$^+$ spectral problem:

$$\psi_x = U(\lambda, q, q^*)\psi \tag{3.4}$$

and the NLS$^-$ spectral problem:

$$\psi_x = U(\lambda, q, -q^*)\psi, \tag{3.5}$$

we have the following lemma.

Lemma 3.1. (1) *If f is a solution of (3.4) with $\lambda = \lambda_1$, then $S_+ f$ is a solution of (3.4) with $\lambda = -\lambda_1^*$; there exists a solution f of (3.4) with $\lambda = \lambda_1$ satisfying $f^{(2)} = f^{(1)*}$ if and only if* $\mathrm{Re}\,\lambda_1 = 0$. (2) *If f is a solution of (3.5) with $\lambda = \lambda_1$, then $S_- f$ is a solution of (3.5) with $\lambda = -\lambda_1^*$; there exists no solution f of (3.5) satisfying $f^{(2)} = f^{(1)*}$ if $q \neq 0$.*

The NLSESCS are reduced from the AKNSESCS defined by

$$\varphi_{j,x} = U(\lambda_j, q, r)\varphi_j, \qquad \varphi'_{j,x} = U(\lambda'_j, q, r)\varphi'_j, \qquad j = 1, \ldots, m, \tag{3.6a}$$

$$\phi_{j,x} = U(\zeta_j, q, r)\phi_j, \qquad j = 1, \ldots, n, \tag{3.6b}$$

$$q_t = -\mathrm{i}(q_{xx} - 2q^2 r) + \sum_{j=1}^{m} \left[\left(\varphi_j^{(1)}\right)^2 + \left(\varphi_j'^{(1)}\right)^2 \right] + \sum_{j=1}^{n} \left(\phi_j^{(1)}\right)^2, \tag{3.6c}$$

$$r_t = \mathrm{i}(r_{xx} - 2qr^2) + \sum_{j=1}^{m} \left[\left(\varphi_j^{(2)}\right)^2 + (\varphi_j'^{(2)})^2 \right] + \sum_{j=1}^{n} \left(\phi_j^{(2)}\right)^2, \tag{3.6d}$$

where $\lambda_1, \ldots, \lambda_n, \lambda'_1, \ldots, \lambda'_n, \zeta_1, \ldots, \zeta_m$ are $2n + m$ distinct constants. The corresponding Lax pair is

$$\psi_x = U(\lambda, q, r)\psi, \qquad \psi_t = V(\lambda, q, r)\psi + \sum_{j=1}^{m} \left[\frac{H(\varphi_j)}{\lambda - \lambda_j} + \frac{H(\varphi'_j)}{\lambda - \lambda'_j} \right] \psi + \sum_{j=1}^{n} \frac{H(\phi_j)}{\lambda - \zeta_j} \psi. \tag{3.7}$$

(1) Reductions to the NLS$^+$ESCS. Let

$$r = q^*, \quad \lambda'_j = -\lambda_j^*, \quad \varphi'_j = \pm S_+ \varphi_j, \quad j = 1, \ldots, m, \tag{3.8a}$$

$$\operatorname{Re}\zeta_j = 0, \qquad \phi_j^{(2)*} = \phi_j^{(1)} \equiv w_j, \quad j = 1, \ldots, n, \tag{3.8b}$$

then equations (3.6) are reduced to the NLS⁺ESCS

$$\varphi_{j,x} = U(\lambda_j, q, q^*)\varphi_j, \qquad j = 1, \ldots, m, \tag{3.9a}$$

$$w_{j,x} = \zeta_j w_j + q w_j^*, \qquad (\operatorname{Re}\zeta_j = 0), \qquad j = 1, \ldots, n, \tag{3.9b}$$

$$q_t = \mathrm{i}(2|q|^2 q - q_{xx}) + \sum_{j=1}^{m}\left[\left(\varphi_j^{(1)}\right)^2 + \left(\varphi_j^{(2)*}\right)^2\right] + \sum_{j=1}^{n} w_k^2. \tag{3.9c}$$

And system (3.7) is reduced to the Lax pair for the NLS⁺ESCS

$$\psi_x = U(\lambda, q, q^*)\psi,$$

$$\psi_t = V(\lambda, q, q^*)\psi + \sum_{j=1}^{m}\left[\frac{H(\varphi_j)}{\lambda - \lambda_j} + \frac{H(S_+\varphi_j)}{\lambda + \lambda_j^*}\right]\psi + \sum_{j=1}^{n}\frac{H((w_j, w_j^*)^T)}{\lambda - \zeta_j}\psi. \tag{3.10}$$

(2) Reductions to the NLS⁻ESCS. Take $n = 0$ in (3.6) and let

$$r = -q^*, \qquad \lambda_j' = -\lambda_j^*, \qquad \varphi_j' = \pm \mathrm{i} S_-\varphi_j, \qquad j = 1, \ldots, m, \tag{3.11}$$

then equations (3.6) with $n = 0$ are reduced to the NLS⁻ESCS

$$\varphi_{j,x} = U(\lambda_j, q, -q^*)\varphi_j, \qquad j = 1, \ldots, m, \tag{3.12a}$$

$$q_t = \mathrm{i}(-2|q|^2 q - q_{xx}) + \sum_{j=1}^{m}\left[\left(\varphi_j^{(1)}\right)^2 - \left(\varphi_j^{(2)*}\right)^2\right]. \tag{3.12b}$$

Correspondingly, system (3.7) with $n = 0$ is reduced to the Lax pair for the NLS⁻ SCS

$$\psi_x = U(\lambda, q, -q^*)\psi, \qquad \psi_t = V(\lambda, q, -q^*)\psi + \sum_{j=1}^{m}\left[\frac{H(\varphi_j)}{\lambda - \lambda_j} - \frac{H(S_-\varphi_j)}{\lambda + \lambda_j^*}\right]\psi. \tag{3.13}$$

We now reduce the Darboux transformations for the AKNSESCS to the NLSESCS. It is easy to verify the following statements.

Lemma 3.2.

(1) Let f and g be two solutions of the NLS⁺ spectral problem $\psi_x = U(\lambda, q, q^*)\psi$ with $\lambda = \mu, \nu$ respectively, and let C be a complex constant wrt x, then we have

$$\sigma(f, S_+g)^* = \sigma(S_+f, g), \qquad \sigma(S_+f, S_+g)^* = \sigma(f, g),$$

$$\sigma(f, S_+f)^* = \sigma(S_+f, f), \qquad \sigma(S_+f, S_+f)^* = \sigma(f, f);$$

$$W_0(\{C, f\}, \{C^*, S_+f\})^* = W_0(\{C, f\}, \{C^*, S_+f\}),$$

$$W_1(\{C, f\}, \{C^*, S_+f\}; S_+g)^* = S_+W_1(\{C, f\}, \{C^*, S_+f\}; g),$$

$$W_2^{(2)}(\{C, f\}, \{C^*, S_+f\}; 0)^* = W_2^{(1)}(\{C, f\}, \{C^*, S_+f\}; 0).$$

Moreover, if g satisfies $g^{(2)} = g^{(1)*}$ ($\Rightarrow \operatorname{Re}\nu = 0$), then

$$W_1^{(2)}(\{C, f\}, \{C^*, S_+f\}; g)^* = W_1^{(1)}(\{C, f\}, \{C^*, S_+f\}; g).$$

(2) Let f and g be two solutions of the NLS$^-$ spectral problem $\psi_x = U(\lambda, q, -q^*)\psi$ with $\lambda = \mu, \nu$ respectively, and let C be a complex constant wrt x, then we have

$$\sigma(f, S_-g)^* = \sigma(S_-f, g), \qquad \sigma(S_-f, S_-g)^* = -\sigma(f, g),$$

$$\sigma(f, S_-f)^* = \sigma(S_-f, f), \qquad \sigma(S_-f, S_-f)^* = -\sigma(f, f),$$

$$W_0(\{C, f\}, \{-C^*, S_-f\})^* = W_0(\{C, f\}, \{-C^*, S_-f\}),$$

$$W_1(\{C, f\}, \{-C^*, S_+f\}; S_-g)^* = S_-W_1(\{C, f\}, \{-C^*, S_-f\}; g),$$

$$W_2^{(2)}(\{C, f\}, \{-C^*, S_-f\}; 0)^* = -W_2^{(1)}(\{C, f\}, \{-C^*, S_-f\}; 0).$$

Using this lemma, we can reduce binary Darboux transformations for the AKNSESCS to binary Darboux transformations for the NLSESCS.

(1) Darboux transformations for the NLS$^+$ESCS. The binary Darboux transformation (2.6) for the AKNSSCS is reduced to a binary Darboux transformation with an arbitrary function for the NLS$^+$ESCS as follows:

Proposition 3.1. *Given a solution $(q, \varphi_1, \ldots, \varphi_m, w_1, \ldots, w_n)$ of the NLS$^+$ESCS (3.9), let $c(t)$ be a real function satisfying $\dot{c}(t) \geqslant 0$, and let f be a solution of the linear system (3.10) with $\lambda = \zeta_{n+1}, \mathrm{Re}\,\zeta_{n+1} = 0$ and satisfy $f^{(1)} = f^{(2)*}$. Define*

$$\bar{\psi} = \psi - \frac{f}{c(t) + \sigma(f, f)}\sigma(f, \psi), \qquad \bar{q} = q - \frac{(f^{(1)})^2}{c(t) + \sigma(f, f)}, \tag{3.14a}$$

$$\bar{\varphi}_j = \varphi_j - \frac{f}{c(t) + \sigma(f, f)}\sigma(f, \varphi_j), \qquad j = 1, \ldots, m, \tag{3.14b}$$

$$\bar{w}_j = w_j - \frac{f^{(1)}}{c(t) + \sigma(f, f)}\sigma(f, (w_j, w_j^*)^T), \qquad j = 1, \ldots, n, \tag{3.14c}$$

$$\bar{w}_{n+1} = \frac{\sqrt{\dot{c}(t)}f^{(1)}}{c(t) + \sigma(f, f)}, \tag{3.14d}$$

then the new variables $\bar{\psi}, \bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_m$ and $\bar{w}_1, \ldots, \bar{w}_{n+1}$ satisfy system (3.10) with n replaced by $n + 1$. Hence $(\bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_n, \bar{w}_1, \ldots, \bar{w}_{m+1})$ is a solution of the NLS$^+$ESCS (3.9) with n replaced by $n + 1$. Moreover, we have

$$|\bar{q}|^2 = |q|^2 - \partial_x^2 \log[c(t) + \sigma(f, f)]. \tag{3.15}$$

The twice repeated binary Darboux transformation for the AKNSESCS can be reduced to a second binary Darboux transformation with an arbitrary function for the NLS$^+$ESCS as follows:

Proposition 3.2. *Given a solution $(q, \varphi_1, \ldots, \varphi_m, w_1, \ldots, w_n)$ of the NLS$^+$ESCS (3.9), let $c(t)$ be an arbitrary complex function, and f be a solution of the linear system (3.10) with $\lambda = \lambda_{m+1}, \mathrm{Re}\,\lambda_{m+1} \neq 0$. Let $\Delta = W_0(\{c, f\}, \{c^*, S_+f\})$, and define*

$$\bar{\psi} = \Delta^{-1}W_1(\{c, f\}, \{c^*, S_+f\}; \psi), \tag{3.16a}$$

$$\bar{q} = q + \Delta^{-1}W_2^{(1)}(\{c, f\}, \{c^*, S_+f\}; 0), \tag{3.16b}$$

$$\bar{\varphi}_j = \Delta^{-1}W_1(\{c, f\}, \{c^*, S_+f\}; \varphi_j), \qquad j = 1, \ldots, m \tag{3.16c}$$

$$\bar{w}_j = \Delta^{-1} W_1^{-1}(\{c, f\}, \{c^*, S_+ f\}; (w_j, w_j^*)^T), \qquad j = 1, \ldots, n, \quad (3.16d)$$

$$\bar{\varphi}_{m+1} = \sqrt{\dot{c}}(c\Delta)^{-1} W_1(\{c, f\}, \{c^*, S_+ f\}; f), \quad (3.16e)$$

then the new variables $\bar{\psi}, \bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m+1}$ *and* $\bar{w}_1, \ldots, \bar{w}_n$ *satisfy system (3.10) with* m *replaced by* $m + 1$. *Hence* $(\bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{n+1}, \bar{w}_1, \ldots, \bar{w}_m)$ *is a solution of the NLS⁺ESCS (3.9) with* m *replaced by* $m + 1$. *Moreover, we have*

$$|\bar{q}|^2 = |q|^2 - \partial_x^2 \log \Delta. \quad (3.17)$$

If we repeat Darboux transformation (3.14) N times and Darboux transformation (3.16) M times, then we have a general multi-times repeated Darboux transformation with $N+M$ arbitrary functions as follows:

Proposition 3.3. *Given a solution* $(q, \varphi_1, \ldots, \varphi_m, w_1, \ldots, w_n)$ *of the NLS⁺ESCS (3.9), let* f_j *be a solution of the linear system (3.10) with* $\lambda = \zeta_{n+j}, \operatorname{Re} \zeta_{n+j} = 0$, *and satisfy* $f_j^{(1)} = f_j^{(2)*}$, $j = 1, \ldots, N$, *and let* g_j *be a solution of the linear system (3.10) with* $\lambda = \lambda_{m+j}, \operatorname{Re} \lambda_{m+j} \neq 0$, $j = 1, \ldots, M$. *Let* $c_j(t)$ *be an arbitrary real function satisfying* $\dot{c}_j(t) \geqslant 0$, $j = 1, \ldots, N$, *and let* $d_j(t)$ *be an arbitrary complex function,* $j = 1, \ldots, M$. *Let* $F_j = \{c_j, f_j\}, G_j = \{d_j, g_j\}, G'_k = \{d_k^*, S_+ g_j\}$, *and* $\Delta = W_0(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M)$, *and define*

$$\bar{\psi} = \Delta^{-1} W_1(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; \psi), \quad (3.18a)$$

$$\bar{q} = q + \Delta^{-1} W_2^{(1)}(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; 0), \quad (3.18b)$$

$$\bar{\varphi}_j = \Delta^{-1} W_1(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; \varphi_j), \qquad j = 1, \ldots, m, \quad (3.18c)$$

$$\bar{\varphi}_{m+j} = \sqrt{\dot{c}_j}(c_j\Delta)^{-1} W_1(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; g_j), \quad j = 1, \ldots, M, \quad (3.18d)$$

$$\bar{w}_j = \Delta^{-1} W_1^{(1)}(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; (w_j, w_j^*)^T), \quad j = 1, \ldots, n, \quad (3.18e)$$

$$\bar{w}_{n+j} = \sqrt{\dot{d}_j}(d_j\Delta)^{-1} W_1(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; f_j), \quad j = 1, \ldots, N, \quad (3.18f)$$

then the new variables $\bar{\psi}, \bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m+M}$ *and* $\bar{w}_1, \ldots, \bar{w}_{n+N}$ *satisfy system (3.10) with* m, n *replaced by* $m + M, n + N$, *respectively. Hence* $(\bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m+M}, \bar{w}_1, \ldots, \bar{w}_{n+N})$ *is a solution of the NLS⁺ESCS (3.9) with* m, n *replaced by* $m + M, n + N$. *Moreover, we have*

$$|\bar{q}|^2 = |q|^2 - \partial_x^2 \log \Delta. \quad (3.19)$$

(2) Darboux transformations for the NLS⁻ESCS. The binary Darboux transformation for the AKNSESCS cannot be reduced to a Darboux transformation for the NLS⁻ESCS. But the two-times Darboux transformation for the AKNSESCS can be reduced to a binary Darboux transformation with an arbitrary function for the NLS⁻ESCS.

Proposition 3.4. *Given a solution* $(q, \varphi_1, \ldots, \varphi_m)$ *of the NLS⁻ESCS (3.12), let* f *be a solution of the linear system (3.13) with* $\lambda = \lambda_{m+1}, \operatorname{Re} \lambda_{m+1} \neq 0$. *Let* $c(t)$ *be an arbitrary complex function,* $\Delta = W_0(\{c, f\}, \{-c^*, S_- f\})$, *and define*

$$\bar{\psi} = \Delta^{-1} W_1(\{c, f\}, \{-c^*, S_- f\}; \psi), \quad (3.20a)$$

$$\bar{q} = q + \Delta^{-1} W_2^{(1)}(\{c, f\}, \{-c^*, S_- f\}; 0), \quad (3.20b)$$

$$\bar{\varphi}_j = \Delta^{-1} W_1(\{c, f\}, \{-c^*, S_- f\}; \varphi_j), \qquad j = 1, \ldots, m \quad (3.20c)$$

$$\bar{\varphi}_{n+1} = \sqrt{\dot{c}}(c\Delta)^{-1} W_1(\{c, f\}, \{-c^*, S_- f\}; f), \quad (3.20d)$$

then the new variables $\bar{\psi}, \bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m+1}$ satisfy system (3.13) with m replaced by $m + 1$, Moreover, we have

$$|\bar{q}|^2 = |q|^2 + \partial_x^2 \log \Delta. \tag{3.21}$$

Repeating the above Darboux transformation N times gives rise to a general N-times repeated binary Darboux transformation with N arbitrary functions for the NLS$^-$ESCS.

Proposition 3.5. *Given a solution $(q, \varphi_1, \ldots, \varphi_m)$ of the NLS$^-$ equations with sources (3.12), let f_j be a solution of the linear system (3.13) with $\lambda = \lambda_{m+j}$, $\mathrm{Re}\,\lambda_{m+j} \neq 0$, $j = 1, \ldots, N$. Let $c_j(t)$ be an arbitrary complex function, $F_j = \{c_j, f_j\}$, $F_j' = \{-c_j^*, S_- f_j\}$, $j = 1, \ldots, N$, $\Delta = W_0(F_1, F_1', \ldots, F_N, F_N')$, and define*

$$\bar{\psi} = \Delta^{-1} W_1(F_1, F_1', \ldots, F_N, F_N'; \psi), \tag{3.22a}$$

$$\bar{q} = q + \Delta^{-1} W_2^{(1)}(F_1, F_1', \ldots, F_N, F_N'; 0), \tag{3.22b}$$

$$\bar{\varphi}_j = \Delta^{-1} W_1(F_1, F_1', \ldots, F_N, F_N'; \varphi_j), \qquad j = 1, \ldots, m \tag{3.22c}$$

$$\bar{\varphi}_{m+j} = \sqrt{\dot{c}_j}(c_j \Delta)^{-1} W_1(F_1, F_1', \ldots, F_N, F_N'; f_j), \qquad j = 1, \ldots, N \tag{3.22d}$$

then the new variables $\bar{\psi}, \bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m+1}$ satisfy system (3.13) with m replaced by $m + N$, and hence $(\bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m+N})$ is a solution of the NLS$^+$ESCS (3.12) with m replaced by $m + N$. Moreover, we have

$$|\bar{q}|^2 = |q|^2 + \partial_x^2 \log \Delta. \tag{3.23}$$

4. Solutions of the NLS equations with sources

This section is devoted to obtaining some examples of the solutions of the NLSESCS by Darboux transformations and the analysis for these solutions. We use subscripts z_R and z_I to indicate the real part and the imaginary part of a complex number z. For $\forall z = |z|\,\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}$ with $\theta \in (-\pi, \pi]$, we define $\sqrt{z} = \sqrt{|z|}\,\mathrm{e}^{\mathrm{i}\theta/2}$.

4.1. Solutions of the NLS$^+$ESCS

We only consider the NLS$^+$ESCS (3.9) with $m = 0$. We start from the NLS$^+$ESCS (i.e., $m = n = 0$)

$$q_t = \mathrm{i}(2|q|^2 q - q_{xx}) \tag{4.1}$$

and its solution

$$q = \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t}, \tag{4.2}$$

where $\rho \in \mathbb{R}_+$ is a constant. We need to solve the linear system

$$\psi_x = U\big(\lambda, \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t}, \rho\,\mathrm{e}^{-2\mathrm{i}\rho^2 t}\big)\psi, \qquad \psi_t = V\big(\lambda, \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t}, \rho\,\mathrm{e}^{-2\mathrm{i}\rho^2 t}\big)\psi. \tag{4.3}$$

The fundamental solution matrix for the linear system (4.4) is

$$\Psi = \begin{pmatrix} \rho\,\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)+\mathrm{i}\rho^2 t} & (\kappa+\lambda)\,\mathrm{e}^{-\kappa(x+2\mathrm{i}\lambda t)+\mathrm{i}\rho^2 t} \\ (\kappa+\lambda)\,\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)-\mathrm{i}\rho^2 t} & -\rho\,\mathrm{e}^{-\kappa(x+2\mathrm{i}\lambda t)-\mathrm{i}\rho^2 t} \end{pmatrix}, \tag{4.4}$$

where $\kappa = \kappa(\lambda)$ satisfies $\kappa^2 = \lambda^2 + \rho^2$.

4.1.1. Solutions of the NLS⁺ESCS with $m = 0$ and $n = 1$. The NLS⁺ESCS with $m = 0$ and $n = 1$ reads

$$w_{1,x} = \mathrm{i}\ell w_1 + q w_1^*, \tag{4.5a}$$

$$q_t = \mathrm{i}(2|q|^2 q - q_{xx}) + w_1^2. \tag{4.5b}$$

where $\ell \neq 0$ is a real constant. Let f be a solution of system (4.3) with $\lambda = \mathrm{i}\ell$ and satisfy $f^{(1)} = f^{(2)*}$, and let $c(t)$ be an arbitrary real function with $\dot{c}(t) \geqslant 0$, then by proposition 3.3, a solution to the equation is given by

$$q = \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t} - \frac{\left(f_1^{(1)}\right)^2}{c(t) + \sigma(f, f)}, \qquad w_1 = \frac{\sqrt{\dot{c}(t)}\, f_1^{(1)}}{c(t) + \sigma(f, f)}. \tag{4.6}$$

Moreover, we have

$$|q|^2 = \rho^2 - \partial_x^2 \log[c(t) + \sigma(f, f)]. \tag{4.7}$$

For the two cases: $\rho > |\ell|$ and $\rho < |\ell|$, formulae (4.6) will give two different classes of solutions respectively: a dark one-soliton solution and a one-positon solution.

(1) Dark one-soliton solution and scattering property. We take $\rho > |\ell|$ and let $\kappa_1 = \kappa(\mathrm{i}\ell)$. We choose $\kappa = \sqrt{\lambda^2 + \rho^2}$, then κ and $\sqrt{\kappa \pm \lambda}$ are analytic at $\lambda = \mathrm{i}\ell$, and $\kappa_1 = \sqrt{\rho^2 - \ell^2} > 0$. Taking into account that the equality $\rho = \sqrt{\kappa - \lambda}\sqrt{\kappa + \lambda}$ holds near $\lambda = \mathrm{i}\ell$, we choose f as

$$f = \left[\Psi\begin{pmatrix}\sqrt{\kappa - \lambda}/\rho \\ 0\end{pmatrix}\right]_{\lambda=\mathrm{i}\ell} = \left.\begin{pmatrix}\sqrt{\kappa - \lambda}\,\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)+\mathrm{i}\rho^2 t} \\ \sqrt{\kappa + \lambda}\,\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)-\mathrm{i}\rho^2 t}\end{pmatrix}\right|_{\lambda=\mathrm{i}\ell} = \begin{pmatrix}\sqrt{\kappa_1 - \mathrm{i}\ell}\,\mathrm{e}^{\kappa_1(x-2\ell t)+\mathrm{i}\rho^2 t} \\ \sqrt{\kappa_1 + \mathrm{i}\ell}\,\mathrm{e}^{\kappa_1(x-2\ell t)-\mathrm{i}\rho^2 t}\end{pmatrix}.$$

Then one finds that $f^{(2)} = f^{(1)*}$. Calculation yields

$$\sigma(f, f) = \frac{1}{2}\begin{vmatrix} f^{(1)} & \partial_{(\mathrm{i}\ell)} f^{(1)} \\ f^{(2)} & \partial_{(\mathrm{i}\ell)} f^{(2)}\end{vmatrix} = \frac{\rho}{2\kappa_1}\,\mathrm{e}^{2\kappa_1(x-2\ell t)}.$$

Let $c(t) = (2\kappa_1)^{-1}\rho\,\mathrm{e}^{2\kappa_1(at+b)}$ with $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ being constants, then formulae (4.6) give a dark one-soliton solution

$$q = \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t} - \frac{2\kappa_1(\kappa_1 - \mathrm{i}\ell)\,\mathrm{e}^{2\kappa_1(x-2\ell t)+2\mathrm{i}\rho^2 t}}{\rho(\mathrm{e}^{2\kappa_1(at+b)} + \mathrm{e}^{2\kappa_1(x-2\ell t)})} = \frac{1 - \mathrm{e}^{-4\mathrm{i}\theta}\,\mathrm{e}^{2\xi}}{1 + \mathrm{e}^{2\xi}}\rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t}, \tag{4.8a}$$

$$w_1 = \sqrt{\frac{a(\kappa_1 - \mathrm{i}\ell)}{\rho}}\,\frac{2\kappa_1\,\mathrm{e}^{\kappa_1(x-2\ell t)+\kappa_1(at+b)+\mathrm{i}\rho^2 t}}{\mathrm{e}^{2\kappa_1(at+b)} + \mathrm{e}^{2\kappa_1(x-2\ell t)}} = \frac{2\sqrt{a}\kappa_1\,\mathrm{e}^{\xi-\mathrm{i}\theta}}{1 + \mathrm{e}^{2\xi}}\,\mathrm{e}^{\mathrm{i}\rho^2 t}, \tag{4.8b}$$

where

$$\xi = \kappa_1[x - (2\ell + a)t - b], \qquad \theta = \frac{1}{2}\arcsin\frac{\ell}{\rho}.$$

By formula (4.7), one obtains

$$|q|^2 = \rho^2 - \partial_x^2 \log(1 + \mathrm{e}^{2\xi}) = \rho^2 - \frac{\kappa_1^2}{\cosh^2\xi}, \tag{4.9}$$

which shows that $|q|^2$ describes the propagation of a dark soliton on the constant background ρ. The soliton is localized around $\xi = 0$, so the location of the soliton is $x(t) = (2\ell + a)t + b$. and the soliton velocity is $2\ell + a$. If $a = 0$, then $w_1 \equiv 0$, and q defined by (4.8) becomes a dark one-soliton solution [27] of the NLS⁺ equation (4.1).

We fix a solution of system (4.3) as

$$\psi_0(x, t; \lambda) = \begin{pmatrix}\rho\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ (\kappa + \lambda)\,\mathrm{e}^{-\mathrm{i}\rho^2 t}\end{pmatrix}\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)}. \tag{4.10}$$

Then a solution of the NLS$^+$ spectral problem

$$\psi_x = U(\lambda, q, q^*)\psi \tag{4.11}$$

with q defined by (4.8) is given by

$$\psi = \psi_0 - \frac{f\sigma(f,\psi_0)}{c(t)+\sigma(f,f)} = \begin{pmatrix} \rho\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ (\kappa+\lambda)\,\mathrm{e}^{-\mathrm{i}\rho^2 t}\end{pmatrix}\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)} - \begin{pmatrix}\sqrt{\kappa_1 - \mathrm{i}\ell}\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ \sqrt{\kappa_1+\mathrm{i}\ell}\,\mathrm{e}^{-\mathrm{i}\rho^2 t}\end{pmatrix}$$

$$\times\frac{\kappa_1\,\mathrm{e}^{2\xi}\,\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)}}{\rho(\lambda-\mathrm{i}\ell)(1+\mathrm{e}^{2\xi})}\times\begin{vmatrix}\sqrt{\kappa_1-\mathrm{i}\ell} & \rho \\ \sqrt{\kappa_1+\mathrm{i}\ell} & \kappa+\lambda\end{vmatrix}$$

$$= \frac{(\rho^2+\mathrm{i}\ell\lambda-\kappa_1\kappa)\,\mathrm{e}^{2\xi}}{\rho^2(\lambda-\mathrm{i}\ell)(1+\mathrm{e}^{2\xi})}\begin{pmatrix}\rho(\kappa_1-\mathrm{i}\ell)\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ -(\kappa+\lambda)(\kappa_1+\mathrm{i}\ell)\,\mathrm{e}^{-\mathrm{i}\rho^2 t}\end{pmatrix}\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)}. \tag{4.12}$$

Based on formulae (4.8), we can analyse the asymptotic features of the dark one-soliton solution. For fixed t, we have

$$q = \begin{cases}\rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t}[1+o(1)], & x\to-\infty, \\ \rho\,\mathrm{e}^{\mathrm{i}(\pi-4\theta)}\,\mathrm{e}^{2\mathrm{i}\rho^2 t}[1+o(1)], & x\to+\infty,\end{cases} \tag{4.13}$$

$$w_1 \to 0, \qquad x\to\pm\infty. \tag{4.14}$$

It is easy to see that q belongs to the class of potentials satisfying the finite density boundary condition [27]

$$q(x,t) = \rho\,\mathrm{e}^{\mathrm{i}\alpha_\pm(t)}[1+o(1)], \qquad x\to\pm\infty, \tag{4.15}$$

where $\alpha_\pm(t)$ are real functions, and $\beta \equiv \frac{1}{2}(\alpha_+(t)-\alpha_-(t))$ is a real constant independent of t. We now define the scattering data for this class of potentials in a similar way to [23].

First, we define $u = q\,\mathrm{e}^{-\mathrm{i}\alpha_-(t)}$, then u satisfies the standard finite density boundary condition

$$u(x,t) = \begin{cases}\rho[1+o(1)], & x\to-\infty, \\ \rho\,\mathrm{e}^{2\mathrm{i}\beta}[1+o(1)], & x\to+\infty.\end{cases} \tag{4.16}$$

Next, we define transmission and reflection coefficients for the NLS$^+$ spectral system

$$\phi_x = \begin{pmatrix}-\lambda & u \\ u^* & \lambda\end{pmatrix}\phi. \tag{4.17}$$

For $u\equiv\rho$, system (4.3) has two linearly independent solutions

$$\begin{pmatrix}\frac{\rho}{\kappa+\lambda} \\ 1\end{pmatrix}\mathrm{e}^{\kappa x}, \qquad \begin{pmatrix}-1 \\ \frac{\rho}{\kappa+\lambda}\end{pmatrix}\mathrm{e}^{-\kappa x},$$

while for $u\equiv\rho\,\mathrm{e}^{2\mathrm{i}\beta}$, system (4.11) has two linearly independent solutions

$$Q(\beta)\begin{pmatrix}\frac{\rho}{\kappa+\lambda} \\ 1\end{pmatrix}\mathrm{e}^{\kappa x}, \qquad Q(\beta)\begin{pmatrix}-1 \\ \frac{\rho}{\kappa+\lambda}\end{pmatrix}\mathrm{e}^{-\kappa x},$$

where $Q(\beta) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\beta}, \mathrm{e}^{-\mathrm{i}\beta})$. We fix a Jost solution ϕ of system (4.11) by imposing the asymptotic property

$$\phi = \begin{pmatrix}\frac{\rho}{\kappa+\lambda} \\ 1\end{pmatrix}\mathrm{e}^{\kappa x}[1+o(1)], \qquad x\to-\infty, \tag{4.18}$$

while the transmission and reflection coefficients $a(\lambda,t)$ and $b(\lambda,t)$ are determined by the asymptotic estimate

$$\phi = a(\lambda,t)Q(\beta)\begin{pmatrix}\frac{\rho}{\kappa+\lambda} \\ 1\end{pmatrix}\mathrm{e}^{\kappa x} + b(\lambda,t)Q(\beta)\begin{pmatrix}-1 \\ \frac{\rho}{\kappa+\lambda}\end{pmatrix}\mathrm{e}^{-\kappa x}, \qquad x\to+\infty. \tag{4.19}$$

We can now calculate the scattering data for the dark one-soliton solution. In this case, we have $u = q\,\mathrm{e}^{-\mathrm{i}\rho^2 t}$ and $\beta = \pi/2 - 2\theta$. Formula (4.12) implies the function ψ has the asymptotic behaviour

$$\psi = \begin{pmatrix} \rho\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ (\kappa+\lambda)\,\mathrm{e}^{-\mathrm{i}\rho^2 t} \end{pmatrix} \mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)}[1+o(1)], \qquad x \to -\infty, \tag{4.20}$$

$$\psi = \frac{\rho^2 + \mathrm{i}\ell\lambda - \kappa_1\kappa}{\rho^2(\lambda - \mathrm{i}\ell)} \begin{pmatrix} \rho(\kappa_1 - \mathrm{i}\ell)\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ -(\kappa+\lambda)(\kappa_1+\mathrm{i}\ell)\,\mathrm{e}^{-\mathrm{i}\rho^2 t} \end{pmatrix} \mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)}[1+o(1)], \qquad x \to +\infty. \tag{4.21}$$

We now take the Jost solution

$$\phi = Q(-\rho^2 t)(\kappa+\lambda)^{-1}\,\mathrm{e}^{-2\mathrm{i}\kappa\lambda t}\psi, \tag{4.22}$$

then we have

$$\phi = \frac{\rho^2 + \mathrm{i}\ell\lambda - \kappa_1\kappa}{\mathrm{i}\rho(\lambda - \mathrm{i}\ell)} Q(\pi/2 - 2\theta) \begin{pmatrix} \frac{\rho}{\kappa+\lambda} \\ 1 \end{pmatrix} \mathrm{e}^{\kappa x}[1+o(1)], \qquad x \to +\infty, \tag{4.23}$$

which implies that

$$a(\lambda, t) = \frac{\rho^2 + \mathrm{i}\ell\lambda - \kappa_1\kappa}{\mathrm{i}\rho(\lambda - \mathrm{i}\ell)}, \qquad b(\lambda, t) = 0. \tag{4.24}$$

The dark one-soliton solution is a reflectionless potential.

(2) One-positon solution and super-reflectionless property. We take $\rho < |\ell|$ and choose $\kappa = (\operatorname{sign}\lambda_I)\,\mathrm{i}\sqrt{-\lambda^2 - \rho^2}$, then κ is analytic at $\lambda = \mathrm{i}\ell$ and $\kappa(\mathrm{i}\ell) = \mathrm{i}k_1$, where $k_1 = (\operatorname{sign}\ell)\sqrt{\ell^2 - \rho^2}$ is a real constant. Choose a periodic solution of system (4.3) with $\lambda = \mathrm{i}\ell$ as

$$f = \left[\Psi \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]_{\lambda=\mathrm{i}\ell} = \left.\begin{pmatrix} \rho\,\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)+\mathrm{i}\rho^2 t} - (\kappa+\lambda)\,\mathrm{e}^{-\kappa(x+2\mathrm{i}\lambda t)+\mathrm{i}\rho^2 t} \\ (\kappa+\lambda)\,\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)-\mathrm{i}\rho^2 t} + \rho\,\mathrm{e}^{-\kappa(x+2\mathrm{i}\lambda t)-\mathrm{i}\rho^2 t} \end{pmatrix}\right|_{\lambda=\mathrm{i}\ell}$$

$$= \begin{pmatrix} \rho\,\mathrm{e}^{\mathrm{i}(\Theta+\rho^2 t)} - \mathrm{i}(k_1+\ell)\,\mathrm{e}^{-\mathrm{i}(\Theta-\rho^2 t)} \\ \mathrm{i}(k_1+\ell)\,\mathrm{e}^{\mathrm{i}(\Theta-\rho^2 t)} + \rho\,\mathrm{e}^{-\mathrm{i}(\Theta+\rho^2 t)} \end{pmatrix}, \tag{4.25}$$

where $\Theta = k_1(x - 2\ell t)$. One finds $f^{(2)} = f^{(1)*}$, and

$$(f^{(1)})^2 = 2\ell(k_1+\ell)[-k_1\ell^{-1}\cos 2\Theta + \mathrm{i}(\sin 2\Theta - \rho\ell^{-1})]\,\mathrm{e}^{2\mathrm{i}\rho^2 t},$$

$$\sigma(f, f) = 2\ell(k_1+\ell)\left[x - 2\left(k_1^2\ell^{-1} + \ell\right)t + \rho(2k_1\ell)^{-1}\cos 2\Theta\right].$$

Choose $c(t) = 2\ell(k_1+\ell)(at+b)$ with $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ being constants, which implies that $\dot{c}(t) \geqslant 0$. Then formulae (4.6) give a one-positon solution

$$q = \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t} - \frac{(f^{(1)})^2}{c(t) + \sigma(f, f)} = \left[\rho + \frac{k_1\ell^{-1}\cos 2\Theta - \mathrm{i}(\sin 2\Theta - \rho\ell^{-1})}{\gamma + \rho(2k_1\ell)^{-1}\cos 2\Theta}\right]\mathrm{e}^{2\mathrm{i}\rho^2 t}, \tag{4.26a}$$

$$w_1 = \frac{\sqrt{\dot{c}(t)}\,f^{(1)}}{c(t) + \sigma(f, f)} = \sqrt{\frac{a}{2}}\,\frac{\sqrt{1 - k_1\ell^{-1}}\,\mathrm{e}^{\mathrm{i}\Theta} - \mathrm{i}\sqrt{1 + k_1\ell^{-1}}\,\mathrm{e}^{-\mathrm{i}\Theta}}{\gamma + (2k_1\ell)^{-1}\rho\cos 2\Theta}\,\mathrm{e}^{\mathrm{i}\rho^2 t}, \tag{4.26b}$$

where

$$\gamma = x + \left[a - 2\left(\ell + k_1^2\ell^{-1}\right)\right]t + b.$$

Formula (4.7) implies

$$|q|^2 = \rho^2 - \partial_x^2 \log[\gamma + (2k_1\ell)^{-1}\rho\cos 2\Theta] = \rho^2 + \frac{1 + \rho^2\ell^{-2} + 2\rho\ell^{-1}(k_1\gamma\cos 2\Theta - \sin 2\Theta)}{[\gamma + \rho(2k_1\ell)^{-1}\cos 2\Theta]^2}. \tag{4.27}$$

When $\rho = a = 0$, we have $k_1 = \ell$ and $w_1 \equiv 0$, and formulae (4.26) degenerate to a solution of the NLS$^+$ equation (4.1)

$$q = -\frac{\mathrm{e}^{-2\mathrm{i}\ell(x-2\ell t)}}{x - 4\ell t + b}, \tag{4.28}$$

which was given in [25].

A solution of the NLS$^+$ spectral problem (4.11) with the potential q defined by (4.26) is

$$\psi = \psi_0 - \frac{f\sigma(f,\psi_0)}{c(t) + \sigma(f,f)} = \begin{pmatrix} \rho\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ (\kappa+\lambda)\,\mathrm{e}^{-\mathrm{i}\rho^2 t} \end{pmatrix} \mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)} - \begin{pmatrix} [\rho\,\mathrm{e}^{\mathrm{i}\Theta} - \mathrm{i}(k_1+\ell)\,\mathrm{e}^{-\mathrm{i}\Theta}]\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ [\mathrm{i}(k_1+\ell)\,\mathrm{e}^{\mathrm{i}\Theta} + \rho\,\mathrm{e}^{-\mathrm{i}\Theta}]\,\mathrm{e}^{-\mathrm{i}\rho^2 t} \end{pmatrix}$$
$$\times \frac{\mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)}}{4\ell(\lambda - \mathrm{i}\ell)(k_1+\ell)[\gamma + (2k_1\ell)^{-1}\rho\cos 2\Theta]} \begin{vmatrix} \rho\,\mathrm{e}^{\mathrm{i}\Theta} - \mathrm{i}(k_1+\ell)\,\mathrm{e}^{-\mathrm{i}\Theta} & \rho \\ \mathrm{i}(k_1+\ell)\,\mathrm{e}^{\mathrm{i}\Theta} + \rho\,\mathrm{e}^{-\mathrm{i}\Theta} & \kappa+\lambda \end{vmatrix}. \tag{4.29}$$

Based on formulae (4.26) and (4.29), we can analyse the basic features of the one-positon solution. Formulae (4.26) imply that for fixed t and $x \to \pm\infty$, we have the asymptotic estimate

$$q\,\mathrm{e}^{-2\mathrm{i}\rho^2 t} = \rho + [k_1\ell^{-1}\cos 2\Theta - \mathrm{i}(\sin 2\Theta - \rho\ell^{-1})]x^{-1}[1 + O(x^{-1})], \tag{4.30}$$

$$w_1\mathrm{e}^{-\mathrm{i}\rho^2 t} = \sqrt{a/2}\big(\sqrt{1 - k_1\ell^{-1}}\,\mathrm{e}^{\mathrm{i}\Theta} - \mathrm{i}\sqrt{1 + k_1\ell^{-1}}\,\mathrm{e}^{-\mathrm{i}\Theta}\big)x^{-1}[1 + O(x^{-1})], \tag{4.31}$$

for all $\rho \in \mathbb{R}_+$. However, the asymptotic behaviour of $|q|^2$ for $\rho = 0$ is different from that for $\rho > 0$. Actually, for $\rho = 0$, we have

$$|q|^2 = x^{-2}[1 + O(x^{-1})],$$

while for $\rho > 0$, we have

$$|q|^2 = \rho^2 + 2k_1\rho\ell^{-1}x^{-1}\cos 2\Theta[1 + O(x^{-1})]. \tag{4.32}$$

Compared to the dark one-soliton solution, the one-positon solution converges to its background slowly.

As a function of x, the potential q and the source w_1 share the same first-order pole $x = x_0(t)$, which is implicitly determined by the equation

$$2k_1\ell[x_0 + \big(a - 2\ell - 2k_1^2\ell^{-1}\big)t + b] = \rho\cos(2k_1x_0 - 4k_1\ell t).$$

The uniqueness of the solution x_0 can easily be proved. Let $\gamma_0(t) = x_0(t) + \big(a - 2\ell - 2k_1^2\ell^{-1}\big)t + b$, then γ_0 satisfies

$$2k_1\ell\gamma_0 = \rho\cos(2k_1[\gamma_0 - \big(a - 2k_1^2\ell^{-1}\big)t - b]).$$

This equation implies that $\gamma_0(t)$ is a periodic function of t with period $\ell\pi\big/\big(2k_1^3\big)$. We define the velocity of a positon as the velocity of its pole. From this definition, the velocity of the positon is

$$v(t) = v(t+T) = \dot{x}_0(t) = \big[2\ell + 2k_1^2\ell^{-1} - a + \dot{\gamma}_0(t)\big],$$

where $T = \pi\big/\big(2k_1^3\big)$, and the average speed of the positon is

$$\frac{1}{T}\int_0^T v(t)\,\mathrm{d}t = \big(2\ell + 2k_1^2\ell^{-1} - a\big).$$

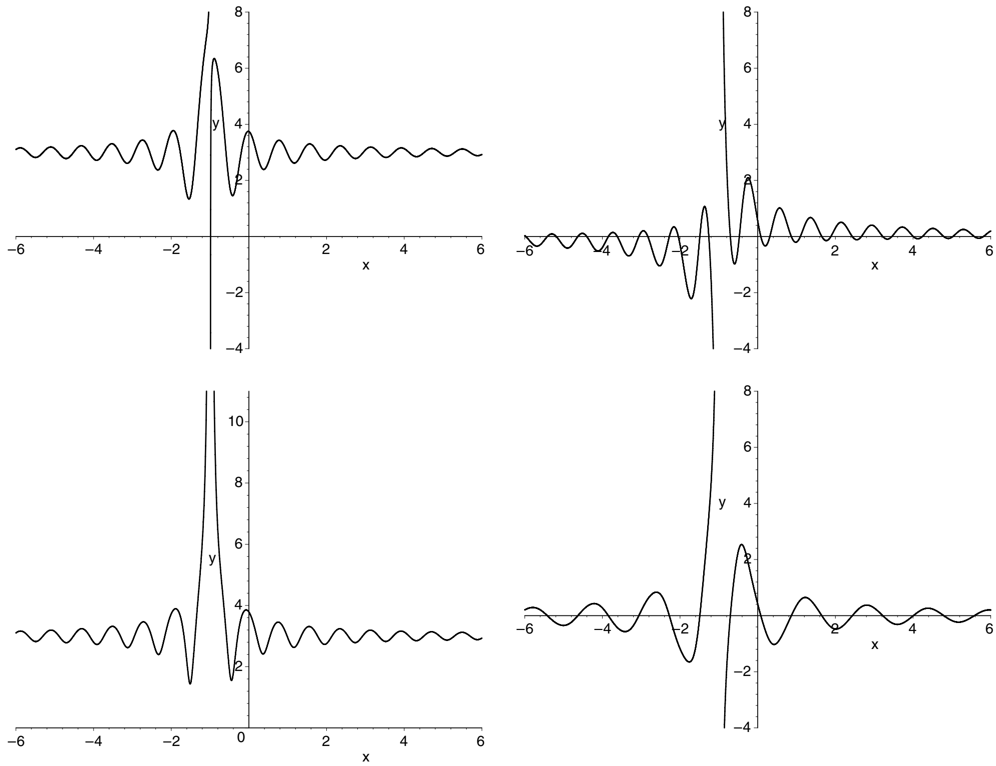

Figure 1. The one-positon solution of the NLS$^+$ESCS (4.5) with $\ell = 5$. The data are $\rho = 3, a = 2$ and $b = 1$. The plots are taken at $t = 0$. The two upper graphs show the real and imaginary parts of q respectively while the two lower graphs show the modulus of q and the real part of w_1 respectively.

In figure 1, we plot a one-positon solution of the NLS$^+$ESCS (4.5).

We now calculate the scattering data for the one-positon solution (4.5). In this case, $u = q\,\mathrm{e}^{-\mathrm{i}\rho^2 t}$ and $\beta = 0$. Formula (4.29) implies the asymptotic behaviour of the function ψ

$$\psi = \begin{pmatrix} \rho\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ (\kappa + \lambda)\,\mathrm{e}^{-\mathrm{i}\rho^2 t} \end{pmatrix} \mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)}[1 + o(1)], \qquad x \to \pm\infty.$$

We take the Jost solution as

$$\phi = Q(-\rho^2 t)(\kappa + \lambda)^{-1}\,\mathrm{e}^{-2\mathrm{i}\kappa\lambda t}\psi,$$

then we have

$$\phi \to \begin{pmatrix} \frac{\rho}{\kappa+\lambda} \\ 1 \end{pmatrix} \mathrm{e}^{\kappa x}, \qquad x \to \pm\infty, \qquad a(\lambda, t) = 1, \qquad b(\lambda, t) = 0.$$

Potentials with reflection coefficient $b = 0$ and transmission coefficient $a = 1$ are called superreflectionless or supertransparent potentials [23]. By this definition, the one-positon solution is superreflectionless.

In [23], positons are defined as long-range analogues of solitons and slowly decreasing, oscillating solutions of nonlinear integrable equations. If we stick to the property of slowly decreasing, the potential q defined by (4.26) should not be called a one-positon solution unless $\rho = 0$. However, we see that other properties such as being the long-range analogue of a soliton and the super-reflectionless property are still valid. Thus it is reasonable to extend the definition of positons as: long-range analogues of solitons, slowly converging, oscillating solutions of nonlinear integrable equations. According to this extended definition, solution (4.26) is a positon solution.

4.1.2. Solutions of the NLS⁺ equation with sources with $m = 0$ and $n = 2$. The NLS⁺ equation with sources with $m = 0$ and $n = 2$ reads

$$w_{1,x} = \mathrm{i}\ell_1 w_1 + q w_1^*, \qquad w_{2,x} = \mathrm{i}\ell_2 w_2 + q w_2^*, \tag{4.33a}$$

$$q_t = \mathrm{i}(2|q|^2 q - q_{xx}) + w_1^2 + w_2^2, \tag{4.33b}$$

where ℓ_1 and ℓ_2 are two distinct real constants. For $j = 1, 2$, let f_j be a solution of system (4.4) with $\lambda = \mathrm{i}\ell_j$ and satisfy $f_j^{(1)} = f_j^{(2)*}$, and let $c_j(t)$ be an arbitrary function with $\dot{c}_j(t) \geqslant 0$. Then by proposition 3.3, a solution of equations (4.33a) is given by

$$q = \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t} + \frac{2\sigma(f_1, f_2) f_1^{(1)} f_2^{(1)} - (c_1(t) + \sigma(f_2, f_2))(f_2^{(1)})^2 - (c_2(t) + \sigma(f_1, f_1))(f_1^{(1)})^2}{(c_1(t) + \sigma(f_1, f_1))(c_2(t) + \sigma(f_2, f_2)) - \sigma(f_1, f_2)^2} \tag{4.34a}$$

$$w_1 = \frac{\sqrt{\dot{c}_1(t)}\big[(c_2(t) + \sigma(f_2, f_2)) f_1^{(1)} - \sigma(f_1, f_2) f_2^{(1)}\big]}{(c_1(t) + \sigma(f_1, f_1))(c_2(t) + \sigma(f_2, f_2)) - \sigma(f_1, f_2)^2}, \tag{4.34b}$$

$$w_2 = \frac{\sqrt{\dot{c}_2(t)}\big[(c_1(t) + \sigma(f_1, f_1)) f_2^{(1)} - \sigma(f_1, f_2) f_1^{(1)}\big]}{(c_1(t) + \sigma(f_1, f_1))(c_2(t) + \sigma(f_2, f_2)) - \sigma(f_1, f_2)^2}. \tag{4.34c}$$

Moreover, we have

$$|q|^2 = \rho^2 - \partial_x^2 \log[(c_1(t) + \sigma(f_1, f_1))(c_2(t) + \sigma(f_2, f_2)) - \sigma(f_1, f_2)^2]. \tag{4.35}$$

For simplicity, we assume $|\ell_1| > |\ell_2|$. According to the three cases for ρ: (i) $\rho > |\ell_j|$, $j = 1, 2$, (ii) $\rho < |\ell_j|$, $j = 1, 2$ and (iii) $|\ell_1| > \rho > |\ell_2|$, formulae (4.34) will give three classes of solutions respectively: dark two-soliton solution, two-positon solution and one-soliton–one-positon solution.

(1) Dark two-soliton solution. For $j = 1, 2$, we take $\rho > |\ell_j|$, and choose

$$f_j = \left[\Psi\begin{pmatrix}\sqrt{\kappa - \lambda}/\rho \\ 0\end{pmatrix}\right]_{\lambda = \mathrm{i}\ell_j} = \begin{pmatrix}\sqrt{\kappa_j - \mathrm{i}\ell_j}\,\mathrm{e}^{\kappa_j(x - 2\ell_j t) + \mathrm{i}\rho^2 t} \\ \sqrt{\kappa_j + \mathrm{i}\ell_j}\,\mathrm{e}^{\kappa_j(x - 2\ell_j t) - \mathrm{i}\rho^2 t}\end{pmatrix},$$

where

$$\kappa = \sqrt{\lambda^2 + \rho^2} \qquad \text{and} \qquad \kappa_j = \sqrt{\rho^2 - \ell_j^2}.$$

Let

$$c_j(t) = \frac{\rho}{2\kappa_j}\,\mathrm{e}^{2\kappa_j(a_j t + b_j)}, \qquad \theta_j = \frac{1}{2}\arcsin\frac{\ell_j}{\rho}, \qquad j = 1, 2,$$

where $a_j \in \mathbb{R}_+$, $b_j \in \mathbb{R}$ are constants, then one finds

$$\sigma(f_j, f_j) = \frac{\rho}{2\kappa_j}\,\mathrm{e}^{2\kappa_j(x - 2\ell_j t)}, \qquad j = 1, 2,$$

$$\sigma(f_1, f_2) = \frac{\rho\sin(\theta_1 - \theta_2)}{\ell_1 - \ell_2}\,\mathrm{e}^{\kappa_1(x - 2\ell_1 t) + \kappa_2(x - 2\ell_2 t)}.$$

Formulae (4.34) yield a dark two-soliton solution

$$q = \frac{1}{\Delta}\begin{vmatrix} \frac{\rho}{2\kappa_1}(1+e^{2\xi_1}) & \frac{\rho\sin(\theta_1-\theta_2)}{\ell_1-\ell_2}e^{\xi_1+\xi_2} & \sqrt{\rho}\,e^{\xi_1+i(\rho^2t-\theta_1)} \\ \frac{\rho\sin(\theta_1-\theta_2)}{\ell_1-\ell_2}e^{\xi_1+\xi_2} & \frac{\rho}{2\kappa_2}(1+e^{2\xi_2}) & \sqrt{\rho}\,e^{\xi_2+i(\rho^2t-\theta_2)} \\ \sqrt{\rho}\,e^{\xi_1+i(\rho^2t-\theta_1)} & \sqrt{\rho}\,e^{\xi_2+i(\rho^2t-\theta_2)} & \rho\,e^{2i\rho^2t} \end{vmatrix}$$

$$= \frac{\rho\,e^{2i\rho^2t}}{\Delta}\begin{vmatrix} \frac{\rho}{2\kappa_1}(1+e^{2\xi_1}) & \frac{\rho\sin(\theta_1-\theta_2)}{\ell_1-\ell_2}e^{\xi_1+\xi_2} & e^{\xi_1-i\theta_1} \\ \frac{\rho\sin(\theta_1-\theta_2)}{\ell_1-\ell_2}e^{\xi_1+\xi_2} & \frac{\rho}{2\kappa_2}(1+e^{2\xi_2}) & e^{\xi_2-i\theta_2} \\ e^{\xi_1-i\theta_1} & e^{\xi_2-i\theta_2} & 1 \end{vmatrix}, \tag{4.36a}$$

$$w_1 = \frac{\sqrt{a_1}\rho\,e^{i\rho^2t}}{\Delta}\begin{vmatrix} \frac{\rho}{2\kappa_2}(1+e^{2\xi_2}) & \frac{\rho\sin(\theta_1-\theta_2)}{\ell_1-\ell_2}e^{\xi_1+\xi_2} \\ e^{\xi_2-i\theta_2} & e^{\xi_1-i\theta_1} \end{vmatrix}, \tag{4.36b}$$

$$w_2 = \frac{\sqrt{a_2}\rho\,e^{i\rho^2t}}{\Delta}\begin{vmatrix} \frac{\rho}{2\kappa_1}(1+e^{2\xi_1}) & \frac{\rho\sin(\theta_1-\theta_2)}{\ell_1-\ell_2}e^{\xi_1+\xi_2} \\ e^{\xi_1-i\theta_1} & e^{\xi_2-i\theta_2} \end{vmatrix}, \tag{4.36c}$$

where

$$\xi_j = \kappa_j[x-(2\ell_j+a_j)t-b_j], \qquad j=1,2,$$

and

$$\Delta = \begin{vmatrix} \frac{\rho}{2\kappa_1}(1+e^{2\xi_1}) & \frac{\rho\sin(\theta_1-\theta_2)}{\ell_1-\ell_2}e^{\xi_1+\xi_2} \\ \frac{\rho\sin(\theta_1-\theta_2)}{\ell_1-\ell_2}e^{\xi_1+\xi_2} & \frac{\rho}{2\kappa_2}(1+e^{2\xi_2}) \end{vmatrix}.$$

(2) Two-positon solutions and positon–positon interaction. For $j=1,2$, we take $\rho<|\ell_j|$, and choose

$$f_j = \left[\Psi\begin{pmatrix}1\\-1\end{pmatrix}\right]_{\lambda=i\ell_j} = \begin{pmatrix}[\rho\,e^{i\Theta_j}-i(k_j+\ell_j)\,e^{-i\Theta_j}]\,e^{i\rho^2t}\\ [i(k_j+\ell_j)\,e^{i\Theta_j}+\rho\,e^{-i\Theta_j}]\,e^{-i\rho^2t}\end{pmatrix},$$

where

$$\kappa = (\operatorname{sign}\lambda_I)\,i\sqrt{-\lambda^2-\rho^2}, \qquad \text{and} \qquad \Theta_j = k_j(x-2\ell_jt), \qquad k_j = (\operatorname{sign}\ell_j)\sqrt{\ell_j^2-\rho^2}.$$

Let

$$c_j(t) = 2\ell_j(k_j+\ell_j)(a_jt+b_j), \qquad \gamma_j = x+\big[a_j-2\big(\ell_j+k_j^2\ell_j^{-1}\big)\big]t+b_j, \qquad j=1,2,$$

where $a_j\in\mathbb{R}_+$, $b_j\in\mathbb{R}$ are constants. Then one finds

$$c_j(t)+\sigma(f_j,f_j) = 2\ell_j(k_j+\ell_j)[\gamma_j+\rho(2k_j\ell_j)^{-1}\cos 2\Theta_j], \qquad j=1,2,$$

$$\sigma(f_1,f_2) = \rho\left(1+\frac{k_1-k_2}{\ell_1-\ell_2}\right)\cos(\Theta_1+\Theta_2)-[\rho^2-(k_1+\ell_1)(k_2+\ell_2)]\frac{\sin(\Theta_1-\Theta_2)}{\ell_1-\ell_2}.$$

Formulae (4.34) give a two-positon solution

$$q = \rho\,e^{2i\rho^2t}+\frac{2f_1^{(1)}f_2^{(1)}\sigma(f_1,f_2)-2\ell_2(k_2+\ell_2)\Gamma_2\big(f_1^{(1)}\big)^2-2\ell_1(k_1+\ell_1)\Gamma_1\big(f_2^{(1)}\big)^2}{4\ell_1\ell_2(k_1+\ell_1)(k_2+\ell_2)\Gamma_1\Gamma_2-\sigma(f_1,f_2)^2}, \tag{4.37a}$$

$$w_1 = \frac{\sqrt{2a_1\ell_1(k_1+\ell_1)}\big[2\ell_2(k_2+\ell_2)\Gamma_2f_1^{(1)}-\sigma(f_1,f_2)f_2^{(1)}\big]}{4(k_1+\ell_1)(k_2+\ell_2)\Gamma_1\Gamma_2-\sigma(f_1,f_2)^2}, \tag{4.37b}$$

$$w_2 = \frac{\sqrt{2a_2\ell_2(k_2+\ell_2)}\big[2\ell_2(k_1+\ell_1)\Gamma_1 f_2^{(1)} - \sigma(f_1, f_2) f_1^{(1)}\big]}{4(k_1+\ell_1)(k_2+\ell_2)\Gamma_1\Gamma_2 - \sigma(f_1, f_2)^2}, \tag{4.37c}$$

where

$$\Gamma_j = \gamma_j + \rho(2k_j\ell_j)^{-1}\cos 2\Theta_j, \qquad j = 1, 2.$$

Assume that $2\ell_1 + 2k_1^2\ell_1^{-1} - a_1 \neq 2\ell_2^2 + 2k_2^2\ell_2^{-1} - a_2$. Fixing γ_1 and letting $t \to \infty$ (which implies $\gamma_2 \to \infty$), we obtain the asymptotic estimate

$$q = \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t} - \frac{k_1\ell_1^{-1}\cos 2\Theta_1 - \mathrm{i}\big(\sin 2\Theta_1 - \rho\ell_1^{-1}\big)}{\gamma_1 + (2k_1\ell_1)^{-1}\rho\cos 2\Theta_1}\,\mathrm{e}^{2\mathrm{i}\rho^2 t}[1 + O(t^{-1})], \tag{4.38a}$$

$$w_1 = \sqrt{\frac{a_1}{2}}\,\frac{\sqrt{1 - k_1\ell_1^{-1}}\,\mathrm{e}^{\mathrm{i}\Theta_1} - \mathrm{i}\sqrt{1 + k_1\ell_1^{-1}}\,\mathrm{e}^{-\mathrm{i}\Theta_1}}{\gamma_1 + (2k_1\ell_1)^{-1}\rho\cos 2\Theta_1}\,\mathrm{e}^{\mathrm{i}\rho^2 t}[1 + O(t^{-1})], \qquad w_2 = O(t^{-1}). \tag{4.38b}$$

Conversely, if we fix γ_2 and let $t \to \infty$, then we obtain

$$q = \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t} - \frac{k_2\ell_2^{-1}\cos 2\Theta_2 - \mathrm{i}\big(\sin 2\Theta_2 - \rho\ell_2^{-1}\big)}{\gamma_2 + (2k_2\ell_2)^{-1}\rho\cos 2\Theta_2}\,\mathrm{e}^{2\mathrm{i}\rho^2 t}[1 + O(t^{-1})], \tag{4.39a}$$

$$w_1 = O(t^{-1}), \qquad w_2 = \sqrt{\frac{a_2}{2}}\,\frac{\sqrt{1 - k_2\ell_2^{-1}}\,\mathrm{e}^{\mathrm{i}\Theta_2} - \mathrm{i}\sqrt{1 + k_2\ell_2^{-1}}\,\mathrm{e}^{-\mathrm{i}\Theta_2}}{\gamma_2 + (2k_2\ell_2)^{-1}\rho\cos 2\Theta_2}\,\mathrm{e}^{\mathrm{i}\rho^2 t}[1 + O(t^{-1})]. \tag{4.39b}$$

Thus we have proved that the two-positon solution decays into two positons asymptotically as $t \to \infty$, and the collision of the two positons is completely insensitive. Even the additional phase shifts in the collision of two dark solitons are absent here.

(3) One-soliton–one-positon solution and soliton–positon interaction. We let ρ satisfy $|\ell_1| < \rho < |\ell_2|$, and choose

$$f_1 = \left[\Psi\begin{pmatrix}\sqrt{\kappa - \lambda}/\rho \\ 0\end{pmatrix}\right]_{\lambda=\mathrm{i}\ell_1} = \begin{pmatrix}\sqrt{\kappa_1 - \mathrm{i}\ell_1}\,\mathrm{e}^{\kappa_1(x-2\ell_1 t)+\mathrm{i}\rho^2 t} \\ \sqrt{\kappa_1 + \mathrm{i}\ell_1}\,\mathrm{e}^{\kappa_1(x-2\ell_1 t)-\mathrm{i}\rho^2 t}\end{pmatrix} = \sqrt{\rho}\,\mathrm{e}^{\kappa_1(x-2\ell_1 t)}\begin{pmatrix}\mathrm{e}^{\mathrm{i}(\rho^2 t-\theta_1)} \\ \mathrm{e}^{-\mathrm{i}(\rho^2 t-\theta_1)}\end{pmatrix},$$

where

$$\kappa = \sqrt{\lambda^2 + \rho^2}, \qquad \kappa_1 = \sqrt{\rho^2 - \ell_1^2}, \qquad \theta_1 = \frac{1}{2}\arcsin\frac{\ell_1}{\rho},$$

and choose

$$f_2 = \left[\Psi\begin{pmatrix}1 \\ -1\end{pmatrix}\right]_{\lambda=\mathrm{i}\ell_2} = \begin{pmatrix}[\rho\,\mathrm{e}^{\mathrm{i}\Theta_2} - \mathrm{i}(k_2+\ell_2)\,\mathrm{e}^{-\mathrm{i}\Theta_2}]\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ [\mathrm{i}(k_2+\ell_2)\,\mathrm{e}^{\mathrm{i}\Theta_2} + \rho\,\mathrm{e}^{-\mathrm{i}\Theta_2}]\,\mathrm{e}^{-\mathrm{i}\rho^2 t}\end{pmatrix},$$

where

$$\kappa = (\operatorname{sign}\operatorname{Im}\lambda)\,\mathrm{i}\sqrt{-\lambda^2 - \rho^2}, \qquad \Theta_2 = k_2(x - 2\ell_2 t), \qquad k_2 = (\operatorname{sign}\ell_2)\sqrt{\ell_2^2 - \rho^2}.$$

Let

$$c_1(t) = \frac{\rho}{2\kappa_1}\,\mathrm{e}^{2\kappa_1(a_1 t + b_1)}, \qquad \xi_1 = \kappa_1[x - (2\ell_1 + a_1)t - b_1],$$

$$c_2(t) = 2\ell_2(k_2+\ell_2)(a_2 t + b_2), \qquad \gamma_2 = x + \big[a_2 - 2\big(\ell_2 + k_2^2\ell_2^{-1}\big)\big]t + b_2,$$

where $a_j \in \mathbb{R}_+$, $b_j \in \mathbb{R}$, $j = 1, 2$, are constants. Then one finds

$$c_1(t) + \sigma(f_1, f_1) = \frac{\rho}{2\kappa_1} \mathrm{e}^{2\kappa_1(a_1 t + b_1)}(1 + \mathrm{e}^{2\xi_1}),$$

$$c_2(t) + \sigma(f_2, f_2) = 2\ell_2(k_2 + \ell_2)[\gamma_2 + (2k_2\ell_2)^{-1}\rho\cos 2\Theta_2],$$

$$\sigma(f_1, f_2) = \frac{\sqrt{\rho}\,\mathrm{e}^{\kappa_1(x - 2\ell_1 t)}}{\ell_2 - \ell_1}[(k_2 + \ell_2)\cos(\theta_1 - \Theta_2) - \rho\sin(\theta_1 + \Theta_2)].$$

Formulae (4.34) give a one-soliton–one-positon solution

$$q = \mathrm{e}^{2\mathrm{i}\rho^2 t}\left[\rho + \frac{2\sqrt{\rho}\,\mathrm{e}^{2\xi_1 - \mathrm{i}\theta_1}AB - 2\ell_2(k_2 + \ell_2)\rho\,\mathrm{e}^{2(\xi_1 - \mathrm{i}\theta_1)}\Gamma_2 - \rho(2\kappa_1)^{-1}(1 + \mathrm{e}^{2\xi_1})A^2}{\rho\kappa_1^{-1}\ell_2(k_2 + \ell_2)(1 + \mathrm{e}^{2\xi_1})\Gamma_2 - \mathrm{e}^{2\xi_1}B^2}\right] \tag{4.40a}$$

$$w_1 = \frac{\sqrt{\rho a_1}[2\ell_2(k_2 + \ell_2)\sqrt{\rho}\,\mathrm{e}^{\xi_1 - \mathrm{i}\theta_1}\Gamma_2 - \mathrm{e}^{\xi_1}AB]\,\mathrm{e}^{\mathrm{i}\rho^2 t}}{\rho\kappa_1^{-1}\ell_2(k_2 + \ell_2)(1 + \mathrm{e}^{2\xi_1})\Gamma_2 - \mathrm{e}^{2\xi_1}B^2} \tag{4.40b}$$

$$w_2 = \frac{\sqrt{2a_2\ell_2(k_2 + \ell_2)}[\rho(2\kappa_1)^{-1}(1 + \mathrm{e}^{2\xi_1})A - \sqrt{\rho}\,\mathrm{e}^{2\xi_1 - \mathrm{i}\theta_1}B]\,\mathrm{e}^{\mathrm{i}\rho^2 t}}{\rho\kappa_1^{-1}\ell_2(k_2 + \ell_2)(1 + \mathrm{e}^{2\xi_1})\Gamma_2 - \mathrm{e}^{2\xi_1}B^2} \tag{4.40c}$$

where

$$\Gamma_2 = \gamma_2 + \rho(2k_2\ell_2)^{-1}\cos 2\Theta_2, \qquad A = \rho\,\mathrm{e}^{\mathrm{i}\Theta_2} - \mathrm{i}(k_2 + \ell_2)\,\mathrm{e}^{-\mathrm{i}\Theta_2},$$

$$B = \frac{\sqrt{\rho}}{\ell_2 - \ell_1}[(k_2 + \ell_2)\cos(\theta_1 - \Theta_2) - \rho\sin(\theta_1 + \Theta_2)].$$

Formula (4.35) implies that

$$|q|^2 = \rho^2 - \partial_x^2 \log\left[\rho\kappa_1^{-1}\ell_2(k_2 + \ell_2)(1 + \mathrm{e}^{2\xi_1})\Gamma_2 - \mathrm{e}^{2\xi_1}B^2\right]. \tag{4.41}$$

It is easy to see that

$$\kappa_1^{-1}\xi_1 - \gamma_2 = \left[2(\ell_2 + k_2^2\ell_2^{-1} - \ell_1) - a_1 - a_2\right]t - b_1 - b_2.$$

Assume

$$2\left(\ell_2 + k_2^2\ell_2^{-1} - \ell_1\right) - a_1 - a_2 > 0.$$

We now fix γ_2, and let $t \to -\infty$ (which implies $\xi_1 \to -\infty$), then we obtain the estimate

$$q = \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t} + \frac{k_2\ell_2^{-1}\cos 2\Theta_2 - \mathrm{i}(\sin 2\Theta_2 - \rho\ell^{-1})}{\gamma_2 + (2k_2\ell_2)^{-1}\rho\cos 2\Theta_2}\,\mathrm{e}^{2\mathrm{i}\rho^2 t}[1 + O(\mathrm{e}^{-2|\xi_1|})], \tag{4.42a}$$

$$w_1 = O(\mathrm{e}^{\xi_1}), \qquad w_2 = \sqrt{\frac{a_2}{2}}\,\frac{\sqrt{1 - k_2\ell_2^{-1}}\,\mathrm{e}^{\mathrm{i}\Theta_2} - \mathrm{i}\sqrt{1 + k_2\ell_2^{-1}}\,\mathrm{e}^{-\mathrm{i}\Theta_2}}{\gamma_2 + (2k_2\ell_2)^{-1}\rho\cos 2\Theta_2}\,\mathrm{e}^{\mathrm{i}\rho^2 t}[1 + O(\mathrm{e}^{-2|\xi_1|})], \tag{4.42b}$$

and

$$|q|^2 = \rho^2 - \partial_x^2 \log[\gamma_2 + \rho(2k_2\ell_2)^{-1}\cos 2\Theta_2][1 + O(\mathrm{e}^{-2|\xi_1|})]. \tag{4.42c}$$

Let $t \to +\infty$, then we obtain the estimate (for simplicity, we only give the estimate for $|q|^2$)

$$|q|^2 = \rho^2 - \partial_x^2 \log[\gamma_2 + \delta_1 + \rho(2k_2\ell_2)^{-1}\cos 2(\Theta_2 + \delta_2)][1 + O(\mathrm{e}^{-2|\xi_1|})],$$

where

$$\delta_1 = -\frac{\kappa_1}{\ell_2(\ell_2 - \ell_1)}, \qquad \delta_2 = \frac{1}{2}\arcsin\frac{2\kappa_1 k_2(\ell_1\ell_2 - \rho^2)}{\rho^2(\ell_2 - \ell_1)^2}.$$

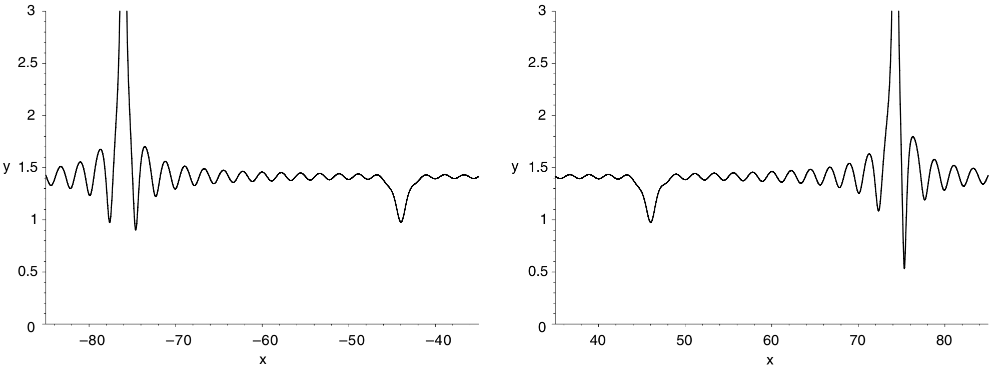

Figure 2. The one-soliton–one-positon solution of the NLS⁺ESCS (4.33*a*) with $\ell_1 = 1$ and $\ell_2 = 2$. The data are $\rho = \sqrt{2}$ and $a_1 = a_2 = b_1 = b_2 = 1$. The two graphs show the modulus of q at $t = -15$ (left) and $t = 15$ (right) respectively.

If we fix ξ_1 and let $t \to \pm\infty$ (which implies $\gamma_2 \to \pm\infty$), then we have the asymptotic estimate

$$q = \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t} - \frac{1 + \mathrm{e}^{-4\mathrm{i}\theta_1}}{1 + \mathrm{e}^{2\xi_1}}\,\mathrm{e}^{2\xi_1 + 2\mathrm{i}\rho^2 t}[1 + O(t^{-1})], \tag{4.43a}$$

$$w_1 = \frac{2\sqrt{a_1}\kappa_1\,\mathrm{e}^{\xi_1 - \mathrm{i}\theta_1}}{1 + \mathrm{e}^{2\xi_1}}\,\mathrm{e}^{\mathrm{i}\rho^2 t}[1 + O(t^{-1})], \qquad w_2 = O(t^{-1}). \tag{4.43b}$$

Thus we have proved that the one-soliton–one-positon solution decays asymptotically into a dark soliton and a positon for large t. The dark soliton recovers completely after the collision with a positon, in other words, a positon is totally transparent to a dark soliton. However, the positon gains phase shifts when colliding with the dark soliton. In figure 2, we plot the one-soliton–one-positon solution.

4.1.3. Solutions of the NLS⁺ESCS with $m = 0$ and $n = N$. The NLS⁺ESCS with $m = 0$ and $n = N$ reads

$$w_{j,x} = \mathrm{i}\ell_j w_j + q w_j^*, \qquad j = 1, \ldots, N, \tag{4.44a}$$

$$q_t = \mathrm{i}(2|q|^2 q - q_{xx}) + \sum_{j=1}^{N} w_j^2, \tag{4.44b}$$

where $\ell_j \neq 0$, $j = 1, \ldots, N$ are N distinct real constants. For $j = 1, \ldots, N$, let f_j be a solution of the system (4.4) with $\lambda = \mathrm{i}\ell_j$ and satisfy $f_j^{(1)} = f_j^{(2)*}$, and let $c_j(t)$ be an arbitrary real function satisfying $\dot{c}_j(t) \geqslant 0$. Then by proposition 3.3, a solution of equations (4.44) is given by

$$q = \rho\,\mathrm{e}^{2\mathrm{i}\rho^2 t} + \frac{\Delta_2}{\Delta_0}, \qquad w_j = \frac{\sqrt{\dot{c}_j(t)}\,\Delta_{1j}}{\Delta_0}, \qquad j = 1, \ldots, N, \tag{4.45}$$

where

$$\Delta_0 = W_0(\{c_1, f_1\}, \ldots, \{c_N, f_N\}), \qquad \Delta_2 = W_2^{(1)}(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; 0),$$

$$\Delta_{1j} = W_1^{(1)}(\{c_1, f_1\}, \ldots, \{c_{j-1}, f_{j-1}\}, \{c_{j+1}, f_{j+1}\}, \ldots, \{c_N, f_N\}; f_j).$$

Moreover, we have

$$|q|^2 = \rho^2 - \partial_x^2 \log \Delta_0. \tag{4.46}$$

For simplicity, we assume $|\ell_1| > \cdots > |\ell_N|$. Then according to the different choice of ρ, we can obtain different classes of solutions.

(1) *Multi-soliton solutions*. We take $\rho > |\ell_j|$, $j = 1, \ldots, N$, and choose

$$c_j(t) = \frac{\rho}{2\kappa_j}\,\mathrm{e}^{2\kappa_j(a_j t + b_j)},$$
$$f_j = \left[\Psi\begin{pmatrix}\sqrt{\kappa - \lambda}/\rho \\ 0\end{pmatrix}\right]_{\lambda = \mathrm{i}\ell_j} = \begin{pmatrix}\sqrt{\kappa_j - \mathrm{i}\ell_j}\,\mathrm{e}^{\kappa_j(x - 2\ell_j t) + \mathrm{i}\rho^2 t} \\ \sqrt{\kappa_j + \mathrm{i}\ell_j}\,\mathrm{e}^{\kappa_j(x - 2\ell_j t) - \mathrm{i}\rho^2 t}\end{pmatrix},$$

where

$$\kappa = \sqrt{\lambda^2 + \rho^2}, \qquad \kappa_j = \sqrt{\rho^2 - \ell_j^2}, \qquad a_j \in \mathbb{R}_+ \qquad \text{and} \qquad b_j \in \mathbb{R},$$

then formulae (4.45) give the dark N-soliton solution.

(2) *Multi-positon solutions*. We take $\rho < |\ell_j|$, $j = 1, \ldots, N$, and choose

$$c_j(t) = 2\ell_j(k_j + \ell_j)(a_j t + b_j),$$
$$f_j = \left[\Psi\begin{pmatrix}1 \\ -1\end{pmatrix}\right]_{\lambda = \mathrm{i}\ell_j} = \begin{pmatrix}[\rho\,\mathrm{e}^{\mathrm{i}\Theta_j} - \mathrm{i}(k_j + \ell_j)\,\mathrm{e}^{-\mathrm{i}\Theta_j}]\,\mathrm{e}^{\mathrm{i}\rho^2 t} \\ [\mathrm{i}(k_j + \ell_j)\,\mathrm{e}^{\mathrm{i}\Theta_j} + \rho\,\mathrm{e}^{-\mathrm{i}\Theta_j}]\,\mathrm{e}^{-\mathrm{i}\rho^2 t}\end{pmatrix},$$

where

$$\kappa = (\operatorname{sign}\operatorname{Im}\lambda)\,\mathrm{i}\sqrt{-\lambda^2 - \rho^2}, \qquad k_j = (\operatorname{sign}\ell_j)\sqrt{\ell_j^2 - \rho^2},$$
$$a_j \in \mathbb{R}_+ \qquad \text{and} \qquad b_j \in \mathbb{R},$$

then formulae (4.45) give the N-positon solution.

(3) *Multi-soliton–multi-positon solutions*. We let ρ satisfy $|\ell_{N_1}| > \rho > |\ell_{N_1+1}|$, where $1 \leqslant N_1 \leqslant N - 1$, and choose

$$c_j(t) = \frac{\rho}{2\kappa_j}\,\mathrm{e}^{2\kappa_j(a_j t + b_j)}, \qquad f_j = \left[\Psi\begin{pmatrix}\sqrt{\kappa - \lambda}/\rho \\ 0\end{pmatrix}\right]_{\lambda = \mathrm{i}\ell_j}, \qquad j = 1, \ldots, N_1,$$

where

$$\kappa = \sqrt{\lambda^2 + \rho^2} \qquad \text{and} \qquad \kappa_j = \sqrt{\rho^2 - \ell_j^2},$$

and

$$c_j(t) = 2\ell_j(k_j + \ell_j)(a_j t + b_j), \qquad f_j = \left[\Psi\begin{pmatrix}1 \\ -1\end{pmatrix}\right]_{\lambda = \mathrm{i}\ell_j}, \qquad j = N_1 + 1, \ldots, N,$$

where

$$\kappa = (\operatorname{sign}\operatorname{Im}\lambda)\,\mathrm{i}\sqrt{-\lambda^2 - \rho^2} \qquad \text{and} \qquad k_j = (\operatorname{sign}\ell_j)\sqrt{\ell_j^2 - \rho^2}.$$

Here $a_j \in \mathbb{R}_+$ and $b_j \in \mathbb{R}$ for $j = 1, \ldots, N$. Then formulae (4.45) give the N_1-soliton–N_2-positon solution ($N_2 = N - N_1$).

4.2. Solutions of the NLS$^-$ESCS

We start from the NLS$^-$ equation without sources

$$q_t = -\mathrm{i}(2|q|^2 q + q_{xx}), \tag{4.47}$$

and its solution

$$q = \rho\, \mathrm{e}^{-2\mathrm{i}\rho^2 t}. \tag{4.48}$$

We need to solve the linear system

$$\psi_x = U(\lambda, \rho\, \mathrm{e}^{-2\mathrm{i}\rho^2 t}, -\rho\, \mathrm{e}^{2\mathrm{i}\rho^2 t})\psi, \qquad \psi_t = V(\lambda, \rho\, \mathrm{e}^{-2\mathrm{i}\rho^2 t}, -\rho\, \mathrm{e}^{2\mathrm{i}\rho^2 t})\psi. \tag{4.49}$$

The fundamental solution matrix for the linear system (4.49) is

$$\Phi = \begin{pmatrix} (\kappa+\lambda)\, \mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)-\mathrm{i}\rho^2 t} & -\rho\, \mathrm{e}^{-\kappa(x+2\mathrm{i}\lambda t)-\mathrm{i}\rho^2 t} \\ -\rho\, \mathrm{e}^{\kappa(x+2\mathrm{i}\lambda t)+\mathrm{i}\rho^2 t} & (\kappa+\lambda)\, \mathrm{e}^{-\kappa(x+2\mathrm{i}\lambda t)+\mathrm{i}\rho^2 t} \end{pmatrix}, \tag{4.50}$$

where $\kappa = \kappa(\lambda)$ satisfies $\kappa^2 = \lambda^2 - \rho^2$.

4.2.1. Solutions of the NLS$^-$ESCS with $n = 1$. The NLS$^-$ESCS with $n = 1$ reads

$$\varphi_{1,x} = U(\lambda_1, q, -q^*)\varphi_1, \tag{4.51a}$$

$$q_t = -\mathrm{i}(2|q|^2 q + q_{xx}) + \left(\varphi_1^{(1)}\right)^2 - \left(\varphi_1^{(2)*}\right)^2, \tag{4.51b}$$

where $\lambda_1 = \lambda_{1R} + \mathrm{i}\lambda_{1I}$ is a complex constant with $\lambda_{1R} > 0, \lambda_{1I} \neq 0$. Let f be a solution of system (4.49) with $\lambda = \lambda_1$, $c(t)$ be an arbitrary complex function, then by proposition 3.4, a solution of equations (4.51) is given by

$$q = \rho\, \mathrm{e}^{-2\mathrm{i}\rho^2 t} + \frac{\Delta_2}{\Delta_0}, \qquad \varphi_1 = \frac{\sqrt{\dot{c}(t)}}{\Delta_0}\begin{pmatrix} \Delta_1^{(1)} \\ \Delta_1^{(2)} \end{pmatrix}, \tag{4.52}$$

where

$$\Delta_0 = \begin{vmatrix} c(t) + \sigma(f,f) & -\frac{|f^{(1)}|^2 + |f^{(2)}|^2}{4\lambda_{1R}} \\ -\frac{|f^{(1)}|^2 + |f^{(2)}|^2}{4\lambda_{1R}} & -c(t)^* - \sigma(f,f)^* \end{vmatrix} = -|c(t) + \sigma(f,f)|^2 - \left(\frac{|f^{(1)}|^2 + |f^{(2)}|^2}{4\lambda_{1R}}\right)^2,$$

$$\Delta_1^{(1)} = \begin{vmatrix} -c(t)^* - \sigma(f,f)^* & -\frac{|f^{(1)}|^2 + |f^{(2)}|^2}{4\lambda_{1R}} \\ -f^{(2)*} & f^{(1)} \end{vmatrix}, \quad \Delta_1^{(2)} = \begin{vmatrix} -c(t)^* - \sigma(f,f)^* & -\frac{|f^{(1)}|^2 + |f^{(2)}|^2}{4\lambda_{1R}} \\ f^{(1)*} & f^{(2)} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} c(t) + \sigma(f,f) & -\frac{|f^{(1)}|^2 + |f^{(2)}|^2}{4\lambda_{1R}} & f^{(1)} \\ -\frac{|f^{(1)}|^2 + |f^{(2)}|^2}{4\lambda_{1R}} & -c(t)^* - \sigma(f,f)^* & -f^{(2)*} \\ f^{(1)} & -f^{(2)*} & 0 \end{vmatrix}.$$

Moreover, we have

$$|q|^2 = \rho^2 + \partial_x^2 \log \Delta_0. \tag{4.53}$$

Topological deformation of the bright one-soliton. We choose f as

$$f = \left[\Phi\begin{pmatrix}1\\0\end{pmatrix}\right]_{\lambda=\lambda_1} = \begin{pmatrix} (\kappa_1 + \lambda_1)\, \mathrm{e}^{-\mathrm{i}\rho^2 t} \\ -\rho\, \mathrm{e}^{\mathrm{i}\rho^2 t} \end{pmatrix} \mathrm{e}^{\kappa_1(x+2\mathrm{i}\lambda_1 t)},$$

where $\kappa_1 = \kappa(\lambda_1)$. Here, we choose $\kappa = \kappa(\lambda) = (\operatorname{sign}\lambda_I)\sqrt{\lambda^2 - \rho^2}$ for Φ defined by (4.50), then κ is analytic at $\lambda = \lambda_1$. Furthermore, under this choice of κ, we have $\lim_{\rho\to 0}\kappa = \lambda$. Calculation yields

$$\sigma(f,f) = \frac{\rho(\kappa_1+\lambda_1)}{2\kappa_1}\,\mathrm{e}^{2\kappa_1(x+2\mathrm{i}\lambda_1 t)}, \qquad |f^{(1)}|^2 = |\kappa_1+\lambda_1|^2\,\mathrm{e}^{2(\kappa_{1R}x - 2\lambda_{1I}t)},$$
$$|f^{(2)}|^2 = \rho^2\,\mathrm{e}^{2(\kappa_{1R}x - 2\lambda_{1I}t)}.$$

We choose $c(t) = (2\kappa_1)^{-1}(\kappa_1+\lambda_1)\,\mathrm{e}^{2(at+b)}$, where a and b are two arbitrary complex numbers, then formulae (4.52) give the topological deformation of the bright one-soliton solution

$$q = \left[\rho + \frac{\frac{\rho^2(\kappa_1+\lambda_1)}{2\kappa_1}\mathrm{e}^{-2\mathrm{i}\eta} - \frac{|\kappa_1+\lambda_1|^2(\kappa_1+\lambda_1)}{2\kappa_1^*}\mathrm{e}^{2\mathrm{i}\eta} + \frac{\rho(\kappa_1+\lambda_1)}{2}\left(\frac{\rho^2}{\kappa_1} + \frac{|\kappa_1+\lambda_1|^2+\rho^2}{\lambda_{1R}} - \frac{|\kappa_1+\lambda_1|^2}{\kappa_1^*}\right)\mathrm{e}^{2\xi}}{\frac{|\kappa_1+\lambda_1|^2}{4|\kappa_1|^2}(\mathrm{e}^{-2\xi} + 2\rho\cos 2\eta + \rho^2\mathrm{e}^{2\xi}) + \left(\frac{|\kappa_1+\lambda_1|^2+\rho^2}{4\lambda_{1R}}\right)^2\mathrm{e}^{2\xi}}\right]\mathrm{e}^{-2\mathrm{i}\rho^2 t}, \tag{4.54a}$$

$$\varphi_1^{(1)} = \sqrt{\frac{a(\kappa_1+\lambda_1)}{\kappa_1}}\cdot\frac{\frac{|\kappa_1+\lambda_1|^2}{2\kappa_1^*}(\mathrm{e}^{-\xi+\mathrm{i}\eta} + \rho\,\mathrm{e}^{\xi-\mathrm{i}\eta}) - \frac{\rho(|\kappa_1+\lambda_1|^2+\rho^2)}{4\lambda_{1R}}\mathrm{e}^{\xi-\mathrm{i}\eta}}{\frac{|\kappa_1+\lambda_1|^2}{4|\kappa_1|^2}(\mathrm{e}^{-2\xi} + 2\rho\cos 2\eta + \rho^2\mathrm{e}^{2\xi}) + \left(\frac{|\kappa_1+\lambda_1|^2+\rho^2}{4\lambda_{1R}}\right)^2\mathrm{e}^{2\xi}}\,\mathrm{e}^{-\mathrm{i}\rho^2 t}, \tag{4.54b}$$

$$\varphi_1^{(2)} = \sqrt{\frac{a(\kappa_1+\lambda_1)}{\kappa_1}}\cdot\frac{\frac{-\rho(\kappa_1^*+\lambda_1^*)}{2\kappa_1^*}(\mathrm{e}^{-\xi+\mathrm{i}\eta} + \rho\,\mathrm{e}^{\xi-\mathrm{i}\eta}) - \frac{(\kappa_1^*+\lambda_1^*)(|\kappa_1+\lambda_1|^2+\rho^2)}{4\lambda_{1R}}\mathrm{e}^{\xi-\mathrm{i}\eta}}{\frac{|\kappa_1+\lambda_1|^2}{4|\kappa_1|^2}(\mathrm{e}^{-2\xi} + 2\rho\cos 2\eta + \rho^2\mathrm{e}^{2\xi}) + \left(\frac{|\kappa_1+\lambda_1|^2+\rho^2}{4\lambda_{1R}}\right)^2\mathrm{e}^{2\xi}}\,\mathrm{e}^{\mathrm{i}\rho^2 t}, \tag{4.54c}$$

where

$$\xi = \kappa_{1R}x - (2\lambda_{1I} + a_R)t - b_R, \qquad \eta = \kappa_{1I}x + (2\lambda_{1R} - a_I)t - b_I.$$

Formula (4.53) implies that

$$|q|^2 = \rho^2 + \partial_x^2 \log\left[4\lambda_{1R}^2|\kappa_1+\lambda_1|^2(\mathrm{e}^{-2\xi} + 2\rho\cos 2\eta + \rho^2\mathrm{e}^{2\xi}) + |\kappa_1|^2(|\kappa_1+\lambda_1|^2+\rho^2)^2\mathrm{e}^{2\xi}\right].$$

When $\rho = 0$, we have $\kappa_1 = \lambda_1$ and the solution given by (4.54) corresponds to the bright one-soliton solution

$$q = -\frac{2\lambda_{1R}\,\mathrm{e}^{2\mathrm{i}\eta_0}}{\cosh 2\xi_0}, \qquad \varphi_1 = \frac{\sqrt{2a\lambda_{1R}}}{\cosh 2\xi_0}\begin{pmatrix}\mathrm{e}^{-\xi_0+\mathrm{i}\eta_0}\\ -\mathrm{e}^{\xi_0-\mathrm{i}\eta_0}\end{pmatrix},$$

where

$$\xi_0 = \lambda_{1R}x - (2\lambda_{1I} + a_R)t - b_R + \log(|\lambda_1|/\sqrt{\lambda_{1R}}), \quad \eta_0 = \lambda_{1I} + (2\lambda_{1R} - a_I)t - b_I + \arg\lambda_1.$$

The topological deformation of the bright one-soliton solution for the NLS$^-$ equation was already known. Here, we have given its correspondence for the NLS$^-$ESCS.

In figure 3, we plot the topological deformation of the bright one-soliton solution.

4.2.2. Solutions of the NLS$^-$ESCS with $n = N$. The NLS$^-$ESCS with $n = N$ reads

$$\varphi_{j,x} = U(\lambda_j, q, -q^*)\varphi_j, \qquad j = 1,\ldots,N, \tag{4.55a}$$
$$q_t = -\mathrm{i}(2|q|^2 q + q_{xx}) + \left(\varphi_1^{(1)}\right)^2 - \left(\varphi_1^{(2)*}\right)^2, \tag{4.55b}$$

where $\lambda_j = \lambda_{jR} + \mathrm{i}\lambda_{jI}$ are distinct complex constants with $\lambda_{jR} > 0, \lambda_{jI} \neq 0$. For $j = 1,\ldots,N$, let

$$F_j = \{c_j, f_j\}, \qquad F_j' = \{-c_j^*, S_- f_j\}, \qquad c_j(t) = \frac{\kappa_j+\lambda_j}{2\kappa_j}\,\mathrm{e}^{a_j t + b_j},$$
$$f_j = \left[\Phi\begin{pmatrix}1\\0\end{pmatrix}\right]_{\lambda=\lambda_j}, \qquad \kappa_j = (\operatorname{sign}\lambda_{jI})\sqrt{\lambda_j^2 - \rho^2},$$

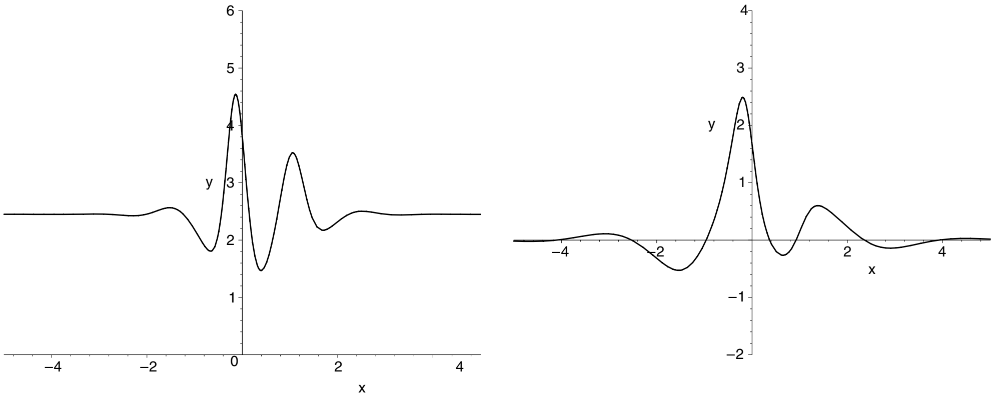

Figure 3. The topological deformation of the bright one-soliton solution of the NLS$^-$ESCS (4.51) with $\lambda_1 = 2 + \mathrm{i}$. The data are $\rho = \sqrt{6}$ and $a_1 = a_2 = b_1 = b_2 = 1$. The two graphs show the modulus of q (left) and the real part of $\phi_1^{(1)}$ (right) at $t = 0$.

then the topological deformation of the bright N-soliton solution of equations (4.55) is given by

$$q = \rho\,\mathrm{e}^{-2\mathrm{i}\rho^2 t} + \frac{\Delta_2}{\Delta_0}, \qquad \varphi_j = \frac{\sqrt{\dot{c}_j(t)}}{\Delta_0}\begin{pmatrix}\Delta_{1j}^{(1)}\\ \Delta_{1j}^{(2)}\end{pmatrix}, \qquad j = 1, \ldots, N,$$

where

$$\Delta_0 = W_0(F_1, F_1', \ldots, F_N, F_N'), \qquad \Delta_2 = W_2^{(0)}(F_1, F_1', \ldots, F_N, F_N'; 0),$$
$$\Delta_{1j}^{(l)} = W_1^{(l)}(F_1, F_1', \ldots, F_{j-1}, F_{j-1}', F_j', F_{j+1}, F_{j+1}', \ldots, F_N, F_N'; f_j),$$
$$l = 1, 2,\; j = 1, \ldots, N.$$

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References

[1] Mel'nikov V K 1992 *Inverse Problems* **8** 133
[2] Vlasov R A and Doktorov E V 1991 *Dokl. Akad. Nauk BSSR* **26** 17
[3] Doktorov E V and Vlasov R A 1993 *Opt. Acta* **30** 223
[4] Nakazawa M, Yomada E and Kubota H 1991 *Phys. Rev. Lett.* **66** 2625
[5] Claude C, Latifi A and Leon J 1991 *J. Math. Phys.* **32** 3321
[6] Mel'nikov V K 1989 *Commun. Math. Phys.* **120** 451
[7] Mel'nikov V K 1989 *Commun. Math. Phys.* **126** 201
[8] Mel'nikov V K 1990 *Inverse Problem* **6** 233
[9] Kaup D J 1987 *Phys. Rev. Lett.* **59** 2063
[10] Leon J and Latifi A 1990 *J. Phys. A: Math. Gen.* **23** 1385
[11] Doktorov E V and Shchesnovich V S 1995 *Phys. Lett.* A **207** 153
[12] Shchesnovich V S and and Doktorov E V 1996 *Phys. Lett.* A **213** 23
[13] Mel'nikov V K 1990 *J. Math. Phys.* **31** 1106
[14] Leon J 1988 *J. Math. Phys.* **29** 2012
Leon J 1990 *Phys. Lett.* A **144** 444
[15] Zeng Y, Ma W and Lin R 2000 *J. Math. Phys.* **41** 5453

[16] Zeng Y 1995 *Acta. Math. Sin.* **15** 337
[17] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[18] Manas M 1996 *J. Phys. A: Math. Gen.* **29** 7721
[19] Zeng Y, Ma W and Shao Y 2001 *J. Math. Phys.* **42** 2113
[20] Zeng Y, Shao Y and Xue W 2003 *J. Phys. A: Math. Gen.* **36** 5035
[21] Xiao T and Zeng Y 2004 *J. Phys. A: Math. Gen.* **37** 7143
[22] Li Y, Gu X and Zou M 1987 *Acta Math. Sin.* **3** 143
[23] Matveev V B 2002 *Theor. Math. Phys.* **131** 483
[24] Rasinariu C and Sukhatme U 1996 *J. Phys. A: Math. Gen.* **29** 1803
[25] Barran S, Kovalyov M and Khare A 1999 *J. Phys. A: Math. Gen.* **32** 6121
[26] Beutler R 1993 *J. Math. Phys.* **34** 3098
[27] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Method in the Theory of Solitons* (Berlin: Springer)